METHODS OF INVESTIGATION OF CHAOTIC MOTIONS IN PERIODICALLY FORCED DYNAMICAL SYSTEMS

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Abstract

Nonlinear systems often exhibit aperiodical behaviour. It is called chaos only if its stochastical properties are caused not by random noise. Our knowledge about these so-called strange attractors is far from being complete.

This work presents a necessary condition for the existence of chaos in addition to methods using simulation on a particular example.

1. Introduction

We know a lot of dynamical systems having strange attractors. Finite parts of the motion can be created by simulation, but there is no powerful analytical method to treat this phenomenon comprehensively, although it is important because aperiodical motion does occur not only in some special systems but it is typical in more than one-degree-of-freedom mechanical systems.

We have chosen a simple one-degree-of-freedom system with strong nonlinearity where chaos may also occur.

Because of the limitations of analytical methods simulation plays more important role in the investigations than usual. Simulation is the creation of a trajectory (of course in case of strange attractors we get only a finite part of it), that is, we take a finite sample from the infinite set of trajectories. This shows the strong limitations of this method: time of simulation, number of required simulations, qualitative

* This research was completed while the author was a graduate student at the Faculty of Mechanical Engineering.

limits. Nowadays it is obvious that the size and complexity of systems to be treated is growing more quickly than the speed and capacity of computers. Thus simulation cannot replace the analytical methods but it can play an important role in guessing rigorous results. Examination of simple systems may provide useful intuition in the observation of complicated phenomena.

2. Mechanical model

The system we are studying is a simple one-degree-of-freedom vibratory system with nonlinear spring. This spring is a Belleville-spring (Fig. 1), [1]. Let F_0 be the force needed to press the spring to plain. If the force R causes a deflection x, the characteristic of this spring is

$$\frac{R}{F_0} = \frac{x}{H}\frac{H^2}{v^2}\left[\left(1-\frac{x}{H}\right)\left(1-\frac{x}{2H}\right)+\frac{v^2}{H^2}\right]. \tag{2.1}$$

See Fig. 2! The mechanical model of this system is

$$m\ddot{x}+k\dot{x}+R(x) = F_e \sin(\omega t), \tag{2.2}$$

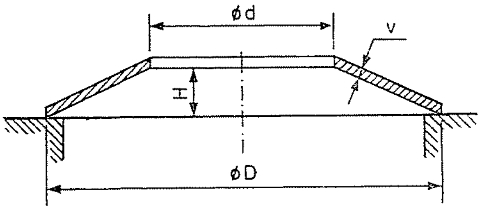


Fig. 1. Belleville-spring

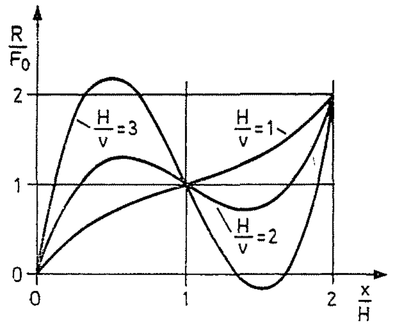


Fig. 2. Characteristic of Belleville-spring

where t denotes the time, ω_e is the frequency of excitation and F_e stands for the exciting force. After transformation of (2.2) the system has four parameters: the relative exciting frequency $\frac{\omega_e}{\alpha}$ where α is the eigenfrequency of the linearized system, the relative damping D ($k=2D\alpha$), the relative excitation $\frac{F_e}{F_0}$ and the measure of nonlinearity $\frac{H}{v}$.

3. Analytical investigations

Using the Poincaré perturbation method [2] we can get diagrams like the one in Fig. 3. This method is unsatisfactory to find chaotic motions.

The first variation system of the Cauchy normal form of the transformed (2.2) can be used for the investigation of the local behaviour of (2.2). If eigenvalues of the matrix of this first variation system are nonpositive in the whole phase space then locally all trajectories tend to each other implying that chaos cannot develop. In the case of the system described by (2.2) this necessary condition of existing of chaos is $\frac{H}{v} > \sqrt{2}$.

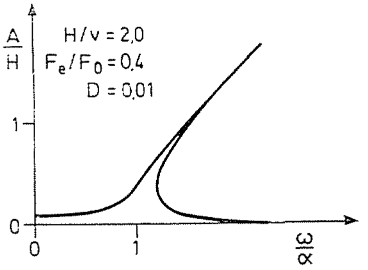


Fig. 3. An amplitude-frequency diagram using the Poincaré perturbation method

4. Numerical results

Results of this section have been obtained by Hamming's predictor-corrector method [3].

4.1 Amplitude-frequency/nonlinearity diagrams

The comparison of the amplitude-frequency diagrams of Figs 6 and 3 shows that the Poincaré method is not powerful enough for our strongly nonlinear case.

Figs 4 and 5 show the effect of different excitations on the weakly nonlinear system. It can be seen that strong excitation causes ultra- and subharmonic resonances (Fig. 5), even the so-called ultra-subharmonic resonance [2] occurs (See Section 4.2, Fig. 10!).

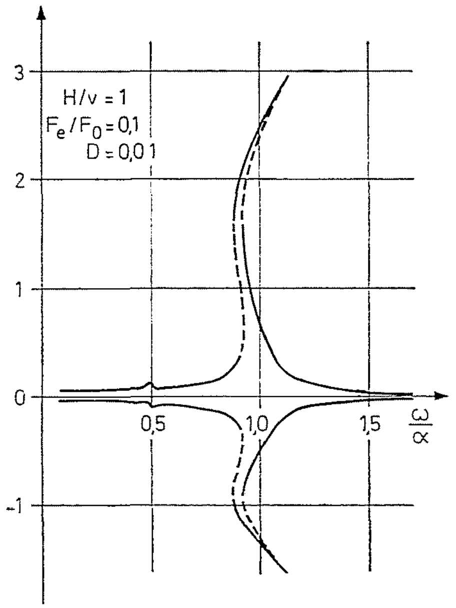


Fig. 4. Amplitude-frequency diagrams

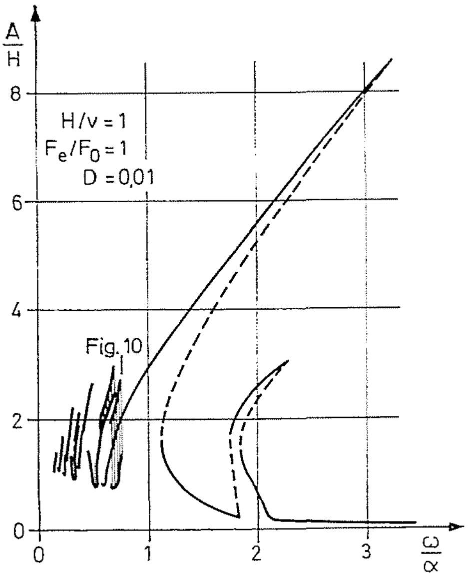


Fig. 5. Amplitude-frequency diagrams

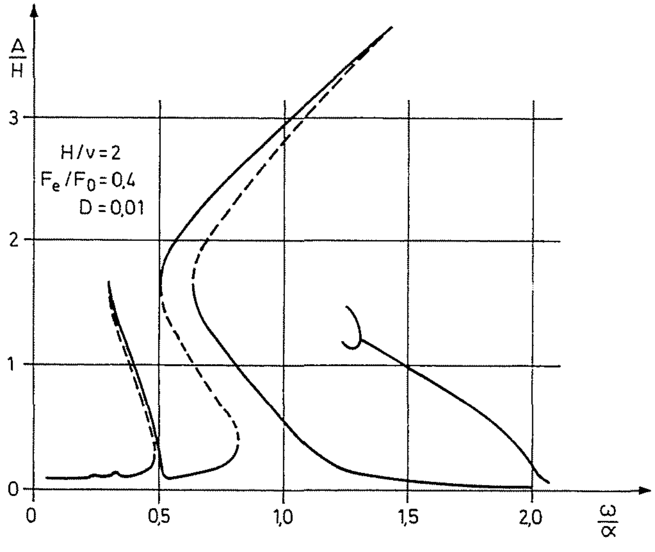


Fig. 6. Amplitude-frequency diagrams

The difference between Figs 7 and 8 is the result of different dampings (See Section 4.2, 4.3!).

In Fig. 9 there are amplitude-nonlinearity diagrams where the limit ($H/v=\sqrt{2}$) coming from the necessary condition obtained by Section 3 is marked.

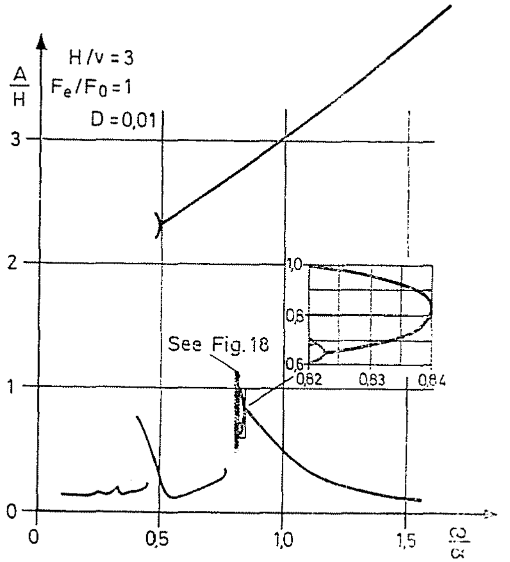


Fig. 7. Amplitude-frequency diagrams

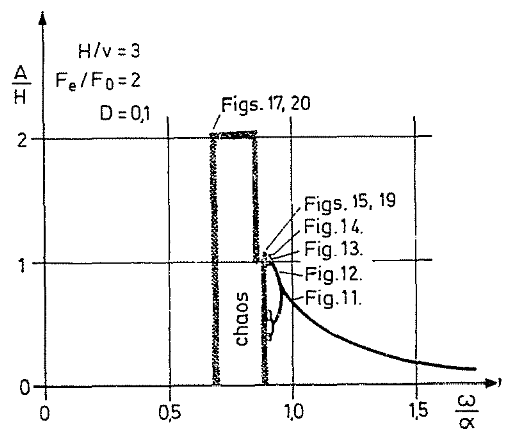


Fig. 8. Amplitude-frequency diagrams

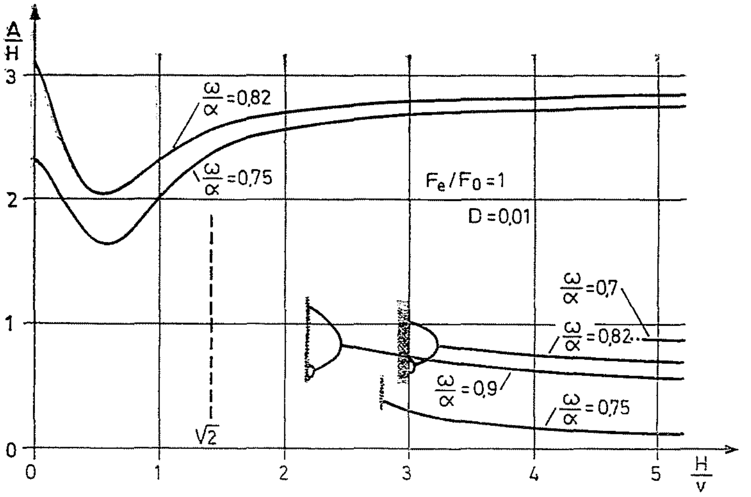


Fig. 9. Amplitude-nonlinearity diagrams

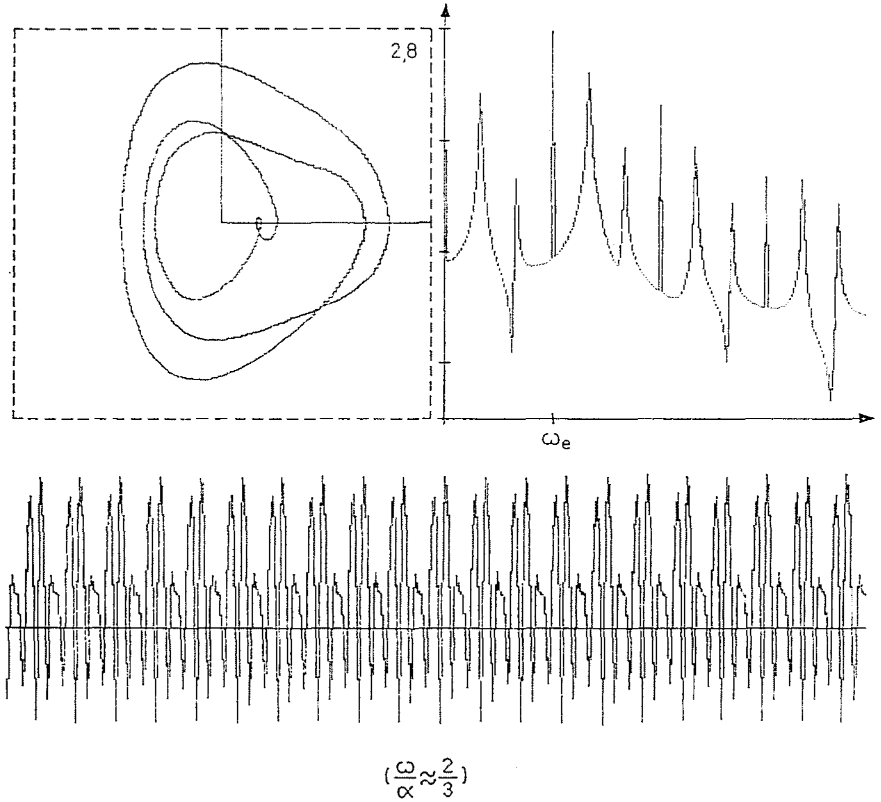
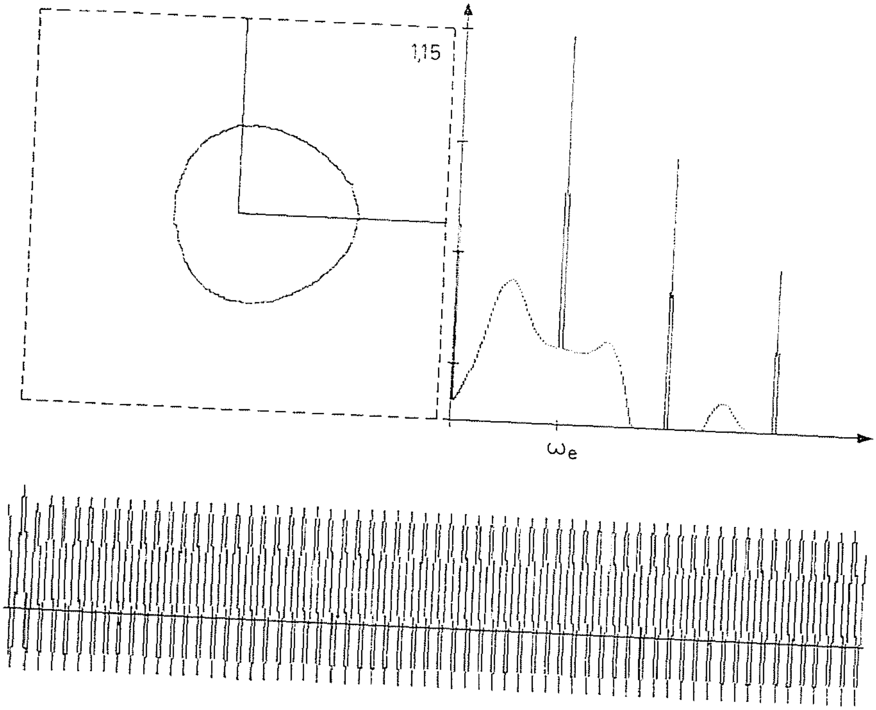
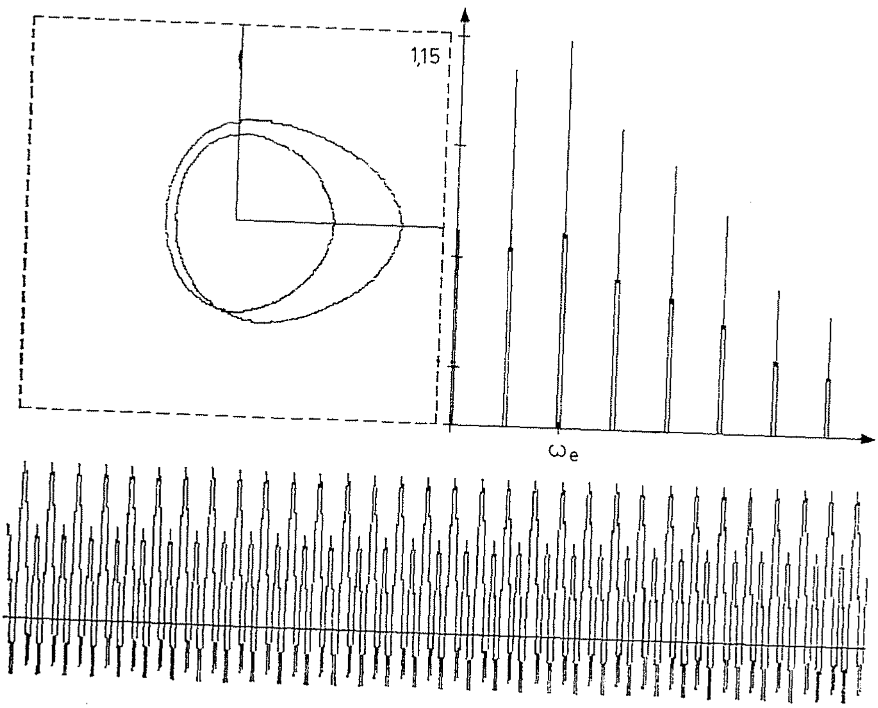


Fig. 10. An ultra-subharmonic diagrams



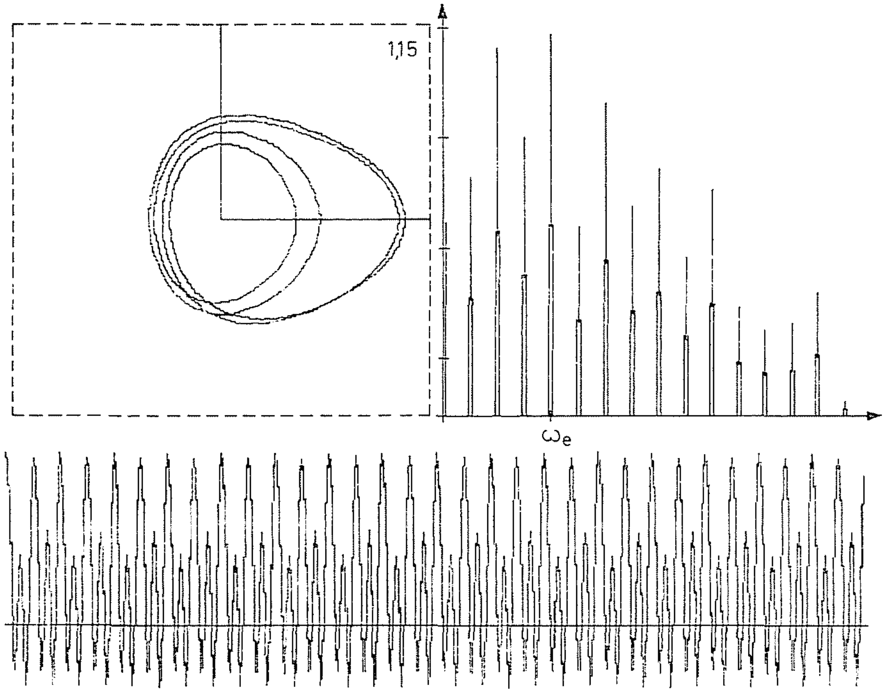
$(\frac{\omega_e}{\alpha} = 1)$

Fig. 11. Transition to chaos — simple-period solution



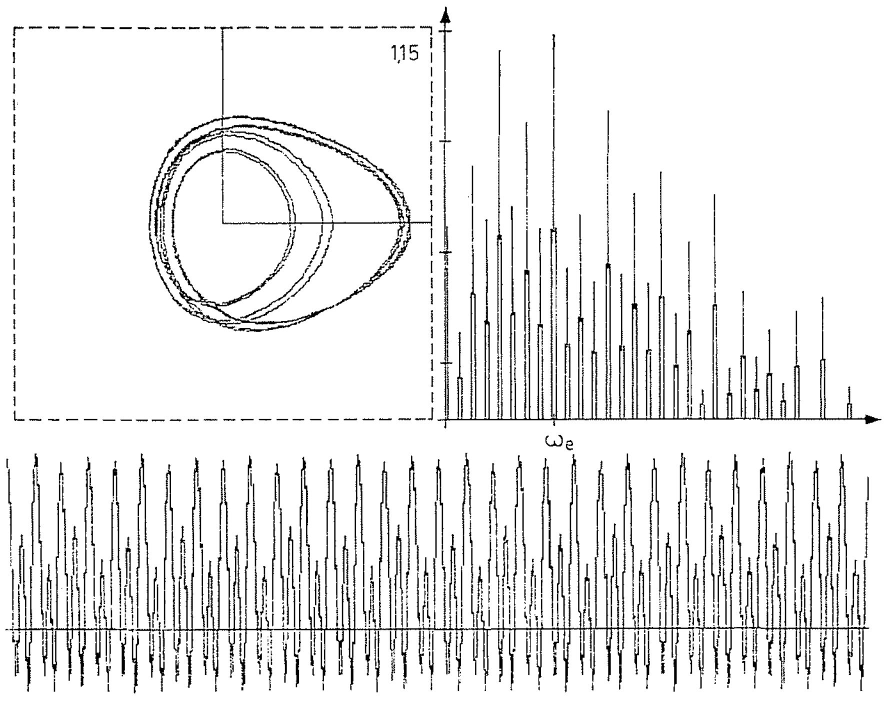
$(\frac{\omega_e}{\alpha} = 0{,}94)$

Fig. 12. Transition to chaos — double-period solution



$\left(\frac{\omega_e}{\alpha} = 0{,}915\right)$

Fig. 13. Transition to chaos — quadcuple-period solution



$\left(\frac{\omega_e}{\alpha} = 0{,}91\right)$

Fig. 14. Transiton to chaos — eightfold-period solution

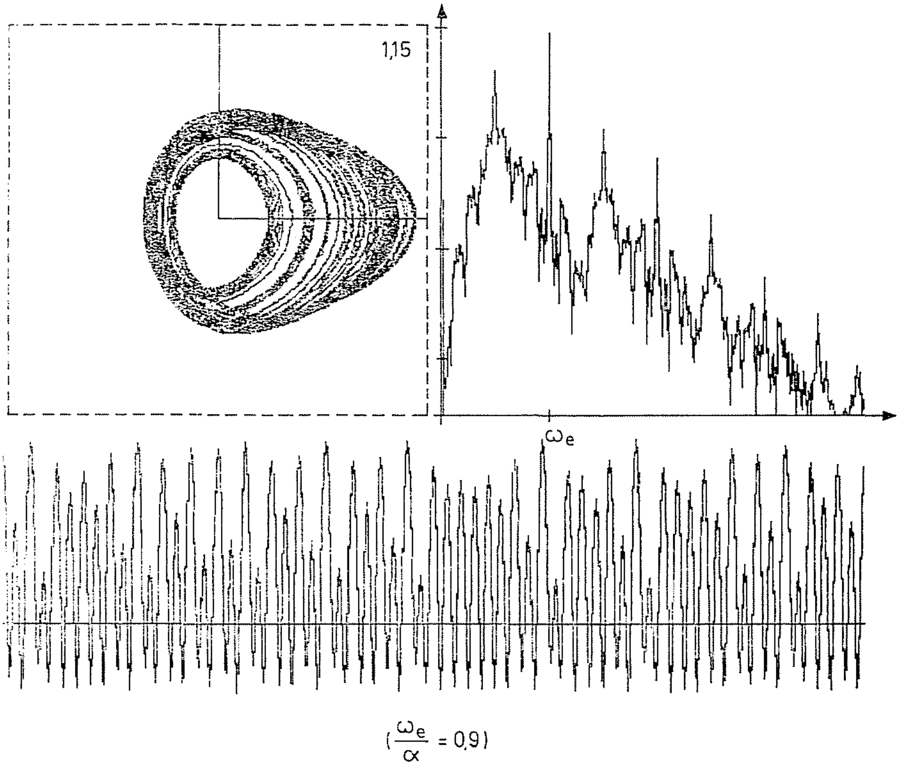


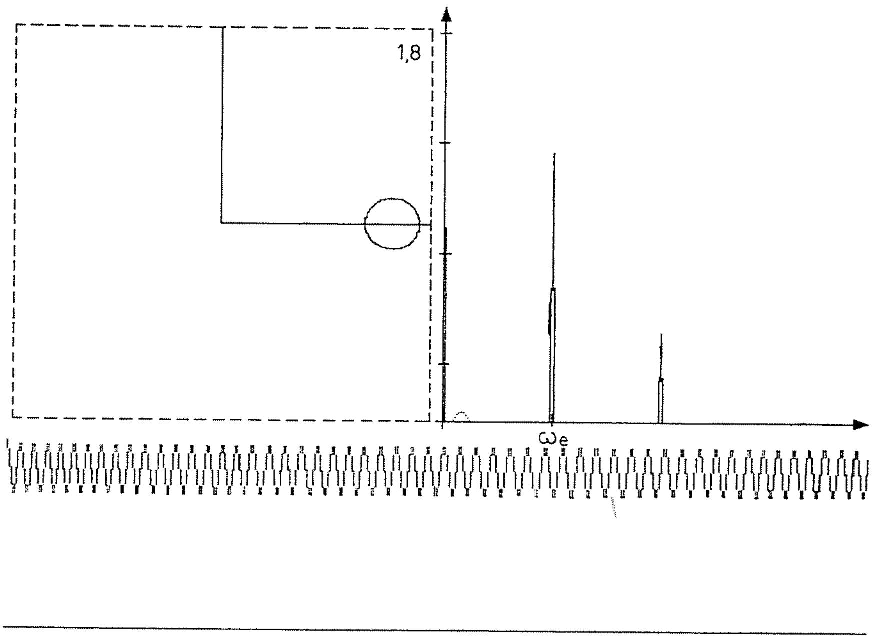
Fig. 15. Weak chaos

4.2. Developing of chaos

Diagrams shown in the previous section do not provide enough information about a particular motion as only the extreme positions (A/H) are shown there (that is where the velocity is equal to zero). In this section particular motions are studied by their time history, trajectory and power spectra.

Before dealing with chaos we present Fig. 10 showing features of an ultra-subharmonic oscillation. In this case $\frac{\omega_e}{\alpha}$ is about 2/3 so the period of this motion is three times larger than the period of the excitation.

In this system cascades of period-doubling bifurcations lead to chaotic motion as shown in Figs 11—15. In this case another interesting fact worth mentioning can be observed. If we decrease the control parameter $\left(\frac{\omega_e}{\alpha}\right)$, a simple periodic



$(\frac{\omega_e}{\alpha} = 0{,}89)$

Fig. 16. Solution around the other equilibrium

solution will be created again but this time it is around the other equilibrium (See Figs 2 and 16!). For an even smaller control parameter chaos develops again but there are important differences between the strange attractors in Figs 15 and 17. In these figures weak and strong chaos, resp., can be seen. In the case of strong chaos (Fig. 17) the system oscillates around both equilibria, and its power spectrum is quite "smooth".

4.3 Poincaré maps

For the investigation of strange attractors of periodically forced dynamical systems we can get clear pictures if we represent the state of the system once in each period, that is, we create a Poincaré map from the differential equation.

We have created Fig. 18 about a strange attractor of Fig. 7 with this method. Similarly we have got the Poincaré maps in Figs 19 and 20 from Figs 15 and 17, resp.

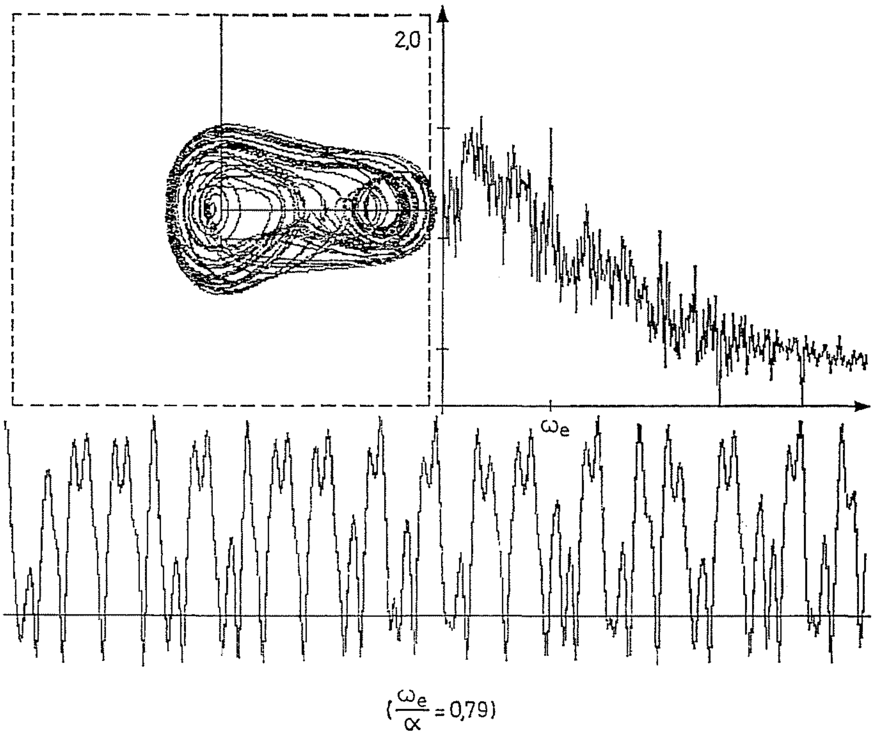


Fig. 17. Strong chaos

In these figures s is the phase of the system where the state is represented in the $s+2\pi n$ ($n=0, 1, 2, \ldots$) seconds. The period of the excitation is 2π because of the transformation of (2.2).

4.4 Cell mapping method

Nonlinear dynamical systems often have two or more attractors simultaneously. Therefore it is very important to know the domains of attraction. There are some analytical methods which can be used in many cases (for example [4]), but in many practical systems we need higher accuracy than the one guaranteed. Cell mapping method [5, 6] is a numerical method for the investigation of the domains of attraction. Drawing the cells we can get a global clear image about the Poincaré map of the system. In Fig. 21 the cell map of the system of Figs 17 and 20 can be seen.

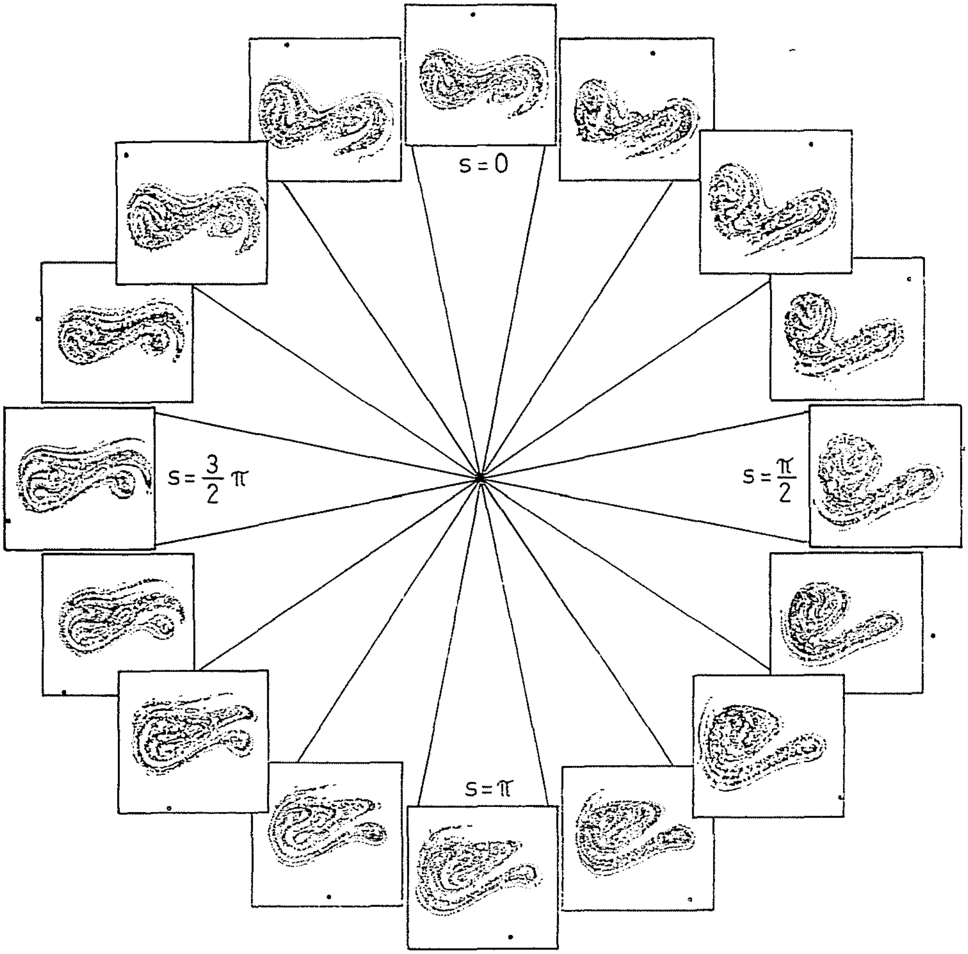


Fig. 18. Poincaré maps of a strange attractor in different times

5. Summary

These investigations demonstrate the fact that the analysis of even simple systems like (2.2) requires many methods. Yet the possibility of two simultaneously existing chaotic motions cannot be excluded (See 4.2!). It is possible that in systems where the nonlinearities are strong enough, weak chaos develops around both equilibria of the spring.

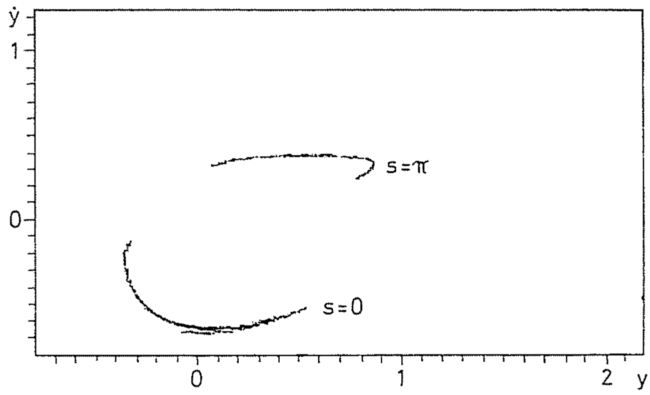


Fig. 19. Poincaré maps of a weak chaos

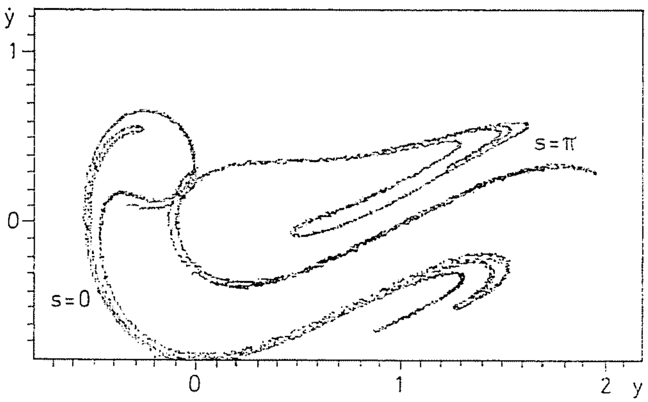


Fig. 20. Poincaré maps of a strong chaos

6. Acknowledgements

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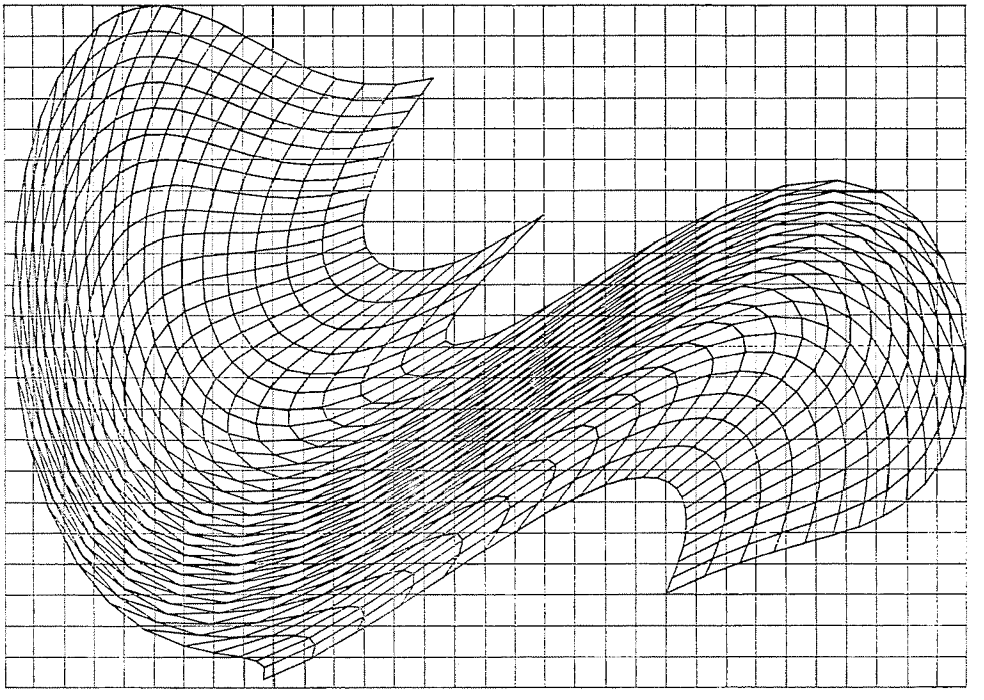

Fig. 21. A cell map

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