CALCULATION OF STATIC AND STATIONARY FIELDS IN INHOMOGENEOUS, ANISOTROPIC MEDIA BY THE VARIATIONAL METHOD

By

I. BÁRDI, O. BIRÓ* and M. GYIMESI*

Department of Theoretical Electricity, Technical University Budapest
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Introduction

In electrodynamics, as known, various physical phenomena can be discussed by identical mathematical methods. By introducing scalar potential, static electric and magnetic fields, and the stationary flow field are calculated by solving the same differential equation. The knowledge of the suitable boundary conditions is also necessary for the solution. In simple cases boundary conditions are Dirichlet or Neumann type. In various cases boundary conditions are Dirichlet type on a part of the surface forming the boundary of the examined region, and Neumann type on other parts of it. This kind of boundary conditions is termed the mixed-type boundary condition. The boundary condition of static fields is mixed type e.g. in the case that the boundary of the examined region is formed by electrodes and by surfaces parallel with electric lines of force.

In the case of stationary current flow the examination of a part of the region of finite conductivity, bounded by electrode surfaces and ideal insulation material, leads to similar boundary conditions.

For the case of homogeneous, isotropic media, variational methods valid for various types of boundary conditions are found in the literature [3], [4], [7]. In this paper the variational calculation will be applied for cases of inhomogeneous, anisotropic, linear media where the boundary condition is mixed type. Elements of the tensor characterizing the material depend on space co-ordinates. On account of space dependence of material characteristics, the partial differential equation of the potential function is not the usual Laplace equation. The method to be described is suited also for the solution of problems where the medium is homogeneous in subregions. In this case the material characteristic is not a continuous function. This case is discussed separately in this paper.

* Undergraduates of electrical engineering.

The main point of the elaborated method is that both the Maxwell equations and the boundary conditions are satisfied in the case of the zero variation of the functional, written in accordance with variation principles for the potential function, connected with the energy of the examined region, or with dissipated power in the case of stationary flow field.

The numerical approximation method of Ritz and Galyerkin, permits the use of a digital computer. Several numerical and programming problems arise in this connection, the description of these, however, is not subject of the present paper.

For demonstration purposes numerical examples are also presented.

Inhomogeneous, anisotropic medium

The solution method is described for the electrostatic field, since this can be formally applied, on the basis of known analogies, also for static magnetic and stationary flow fields.

The solution of Maxwell equations valid in electrostatics, by introducing scalar potential φ, results in solving the partial differential equation

$$\mathrm{div}\,(\epsilon\ \mathrm{grad}\ \varphi) = 0\,. \tag{1}$$

The boundary conditions are

$$\varphi(S_1) = \Phi_{S1}\,, \tag{2}$$

$$n^{+}\ \epsilon\ \mathrm{grad}\ \varphi/_{S2} = 0 \tag{3}$$

where ϵ is the permittivity tensor of the inhomogeneous medium; S_1 denotes the surface parts bounding the examined region where the potential value corresponds to the prescribed function Φ_{S_1}, while S_2 those where the normal component of the displacement vector is zero; n is the normal unit vector to the surface, with positive direction indicating outward; the symbol $^{+}$ denotes the transpose (Fig. 1).

In the knowledge of scalar potential, intensity of electric field and displacement vector can be calculated.

$$E = -\ \mathrm{grad}\ \varphi, \tag{4}$$

$$D = \epsilon\, E\,. \tag{5}$$

Eq. (1) to be solved will be the usual Laplace equation in the case of homogeneous isotropic medium (ϵ is a scalar value independent of co-ordinates). In the case of inhomogeneous, anisotropic medium (ϵ is a tensor with co-ordinates dependent elements), however, the first derivatives of potential also occur in the differential equation. Hence, the methods known for the solution of the Laplace equation cannot be applied.

According to boundary conditions (2) and (3), on one part of the surface bounding the examined region, the potential function is given, while on other parts the normal component of the displacement vector must be zero. This problem belongs to the group of mixed-type boundary conditions.

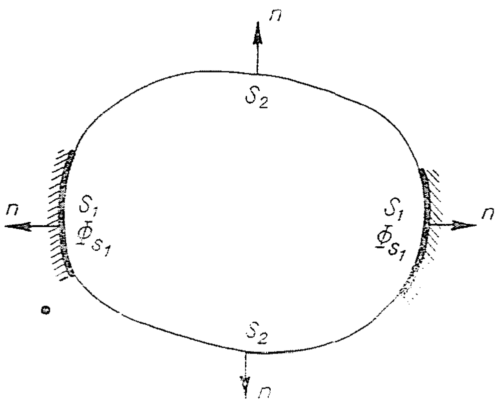

Fig. 1

As known, if the Maxwell equations and the given boundary conditions are satisfied, the energy of the electric or magnetic field has a minimum value, by virtue of Thomson's theorem. Therefore if a functional can be found for the potential function which gives the energy of the examined region if the boundary conditions are satisfied, at its minimum both the Maxwell equations and the boundary conditions are satisfied. It can be proved that the functional

$$W(\varphi) = \frac{1}{2}\iiint\limits_{V} [\operatorname{grad}\varphi^{+}\,\epsilon\operatorname{grad}\varphi]\,\mathrm{d}V + \iint\limits_{S_1} [(\Phi_{S1}-\varphi)\,n^{+}\,\epsilon\operatorname{grad}\varphi]\cdot\mathrm{d}S\,, \quad (6)$$

of energy character has these characteristics. Namely the first member of the functional is the energy of the region, while the second member is zero if boundary conditions are satisfied: to the second a physical meaning can also be given. The second member of the functional is the energy of a double layer which is arranged on surface S_1 and has the moment $\nu = (\Phi_{S1} - \varphi)\,\epsilon$. It should be noted that in the general interpretation, functional (6) has not an extremal but a stationary function. Nevertheless, in the following the expression of the extreme or minimum value of the functional will be used in place of the stationary function. In the course of calculations, namely, exclusively the fact that the first variation is zero will be used.

It will be tried to find the minimum of functional (6) by using variational calculation. It will be proven that an extreme (minimum) value can arise only if Maxwell equations (1) and boundary conditions (2) and (3) are satisfied simultaneously. According to variational calculation, the necessary condition of the existence of a minimum is

$$\delta W(\varphi) = 0 , \tag{7}$$

where δ denotes the first variation. From the first variation of functional (6):

$$\begin{aligned}\delta W(\varphi) = \frac{1}{2} \iiint\limits_V [\operatorname{grad} \eta^+ \epsilon \operatorname{grad} \varphi + \operatorname{grad} \varphi^+ \epsilon \operatorname{grad} \eta] \,\mathrm{d}V + \\ + \iint\limits_{S_1} [-\eta\, n^+ \epsilon \operatorname{grad} \varphi + (\Phi_{S1} - \varphi)\, n^+ \epsilon \operatorname{grad} \eta] \,\mathrm{d}S = 0 ,\end{aligned} \tag{8}$$

where η is an arbitrary, continuously derivable function. Supposing that tensor ϵ is symmetrical (according to [5]), further using the relationship:

$$\operatorname{div} (\eta\, \epsilon \operatorname{grad} \varphi) = \eta \operatorname{div} (\epsilon \operatorname{grad} \varphi) + \operatorname{grad} \varphi^+ \epsilon \operatorname{grad} \varphi , \tag{9}$$

and applying the Gauss theorem, the first variation can be written in the following form:

$$\begin{aligned}\delta W(\varphi) = - \iiint\limits_V [\eta \operatorname{div} (\epsilon \operatorname{grad} \varphi)] \,\mathrm{d}V + \oint\!\!\!\oint\limits_S [\eta n^+ \epsilon \operatorname{grad} \varphi) \,\mathrm{d}S + \\ + \iint\limits_{S_1} [- \eta\, n^+ \epsilon \operatorname{grad} \varphi + (\Phi_{S1} - \varphi)\, n^+ \epsilon \operatorname{grad} \eta] \,\mathrm{d}S = 0 ,\end{aligned} \tag{10}$$

where S denotes the closed surface bounding volume V. Taking into consideration that:

$$\begin{aligned}\oint\!\!\!\oint\limits_S [\eta n^+ \epsilon \operatorname{grad} \varphi) \,\mathrm{d}S = \iint\limits_{S_1} [\eta n^+ \epsilon \operatorname{grad} \varphi] \,\mathrm{d}S + \\ + \iint\limits_{S_2} [\eta n^+ \epsilon \operatorname{grad} \varphi] \,\mathrm{d}S ,\end{aligned} \tag{11}$$

expression (10) can be reduced:

$$\begin{aligned}\delta W(\varphi) = - \iiint\limits_V [\operatorname{div} (\epsilon \operatorname{grad} \varphi)] \,dV + \iint\limits_{S_1} [(\Phi_{S1} - \varphi)\, n^+ \epsilon \operatorname{grad} \eta] \,\mathrm{d}S + \\ + \iint\limits_{S_2} [\eta n^+ \epsilon \operatorname{grad} \varphi] \,\mathrm{d}S = 0 .\end{aligned} \tag{12}$$

Since η and grad η are arbitrary functions which are not identically zero at the boundary of the region, therefore independently of function η, (12) can be zero if and only if the equations

$$\operatorname{div}(\epsilon \operatorname{grad} \varphi) = 0 , \tag{13}$$

$$\varphi(S_1) = \Phi_{S1} , \tag{14}$$

and

$$n^{+} \epsilon \operatorname{grad} \varphi/_{S2} = 0 \tag{15}$$

are satisfied. Among these, (13) is the differential equation to be solved, (14) and (15) are boundary conditions. We have herewith proved that looking for the extremal value of functional (6) is equivalent to solving the Maxwell equations with the given boundary conditions.

Similar relationships can be obtained in the case of two-dimensional problems, in place of the volume integral, however, the surface integral along the planar section is figuring, further in place of the surface integral the integral along the curve bounding the plane region. In this case the normal vector is the normal of the curve, with positive direction indicating outwards.

Consideration of material characteristics continuous in subregions

In our calculations no restriction has so far been made regarding continuity of functions. According to physical considerations, potential function should be continuous, except where there is also a double layer in the examined region. This case, however, will not be discussed. But tensor ϵ may have a break in function of the place co-ordinates. A special case is the medium having constant permittivity in subregions. It can be followed up in the proof of the solution by the variational method that in such a case the previous method can be applied, with the exclusion of break places. Exclusion of break places means the decomposition of integrals to regions where elements of tensor ϵ are continuous. It should be noted that in this case, if potential function is looked for in the set of continuously derivable functions, the solution will only be approximative since the derivative of the potential has a break. Selecting a suitable approximation method, the approximation is converging to the exact solution by "rounding off" the break point. It is advisable, however, to elaborate a method taking into consideration also the break of the derivative of the potential function.

Suppose to exist in the examined region, either side of surface S_3 inhomogeneous anisotropic media of permittivities ϵ_1, and ϵ_2 (Fig. 2).

In symbols of surfaces bounding the subregion, the first subscript denotes the number of the subregion. The second subscript is 1 if it denotes a surface along which potential has a prescribed value, and 2 for a surface

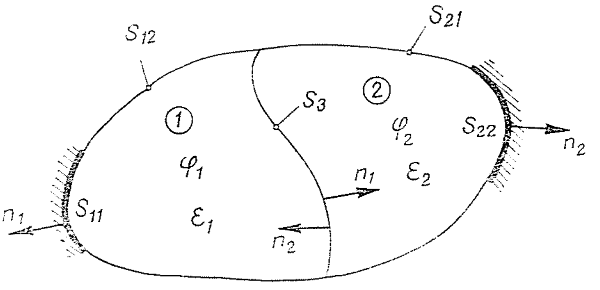

Fig. 2

with zero displacement vector along. Try to find the solution of the Maxwell equations in subregion 1 by using potential function φ_1, while in subregion 2 by using potential function φ_2. Functions φ_1 and φ_2 have to satisfy the Maxwell equations and the boundary conditions. These are given as follows:

$$\operatorname{div}(\epsilon_1 \operatorname{grad} \varphi_1) = 0\,, \tag{16}$$

$$\operatorname{div}(\epsilon_2 \operatorname{grad} \varphi_2) = 0\,, \tag{17}$$

$$\varphi_1(S_{11}) = \Phi_{S11}, \tag{18}$$

$$n_1^+ \epsilon_1 \operatorname{grad} \varphi_1/_{S_{12}} = 0\,, \tag{19}$$

$$\varphi_2(S_{21}) = \Phi_{S_{21}} \tag{20}$$

$$n_2^+ \epsilon_2 \operatorname{grad} \varphi_2/_{S_{22}} = 0\,, \tag{21}$$

$$\varphi_1(S_3) = \varphi_2(S_3)\,, \tag{22}$$

$$n_1^+ \epsilon_1 \operatorname{grad} \varphi_1/_{S_3} + n_2^+ \epsilon_2 \operatorname{grad} \varphi_2/_{S_3} = 0\,. \tag{23}$$

(16) and (17) are partial differential equations originating from the Maxwell equations. Conditions (18), (19), (20), (21) depend on whether the surface bounding the subregion is electrode or force line. Condition (22) specifies that the potential function is continuous along surface S_3. (23) specifies the continuity of the normal component of the displacement vector.

The solution of Eqs. (16) to (23) will be proven to be equivalent to the minimization of the functionals $W_1(\varphi_1, \varphi_2)$ and $W_2(\varphi_1, \varphi_2)$ with respect to functions φ_1 and φ_2, respectively.

$$W_1(\varphi_1, \varphi_2) = \frac{1}{2} \iiint_{V_1} [\operatorname{grad} \varphi_1^+ \epsilon_1 \operatorname{grad} \varphi_1]\, dV + \iint_{S_{11}} [(\Phi_{S11} - \tag{24}$$

$$- \varphi_1)\, n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 \cdot dS + \frac{1}{2} \iint_{S} [(\varphi_2 - \varphi_1)\, n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 + \varphi_1 n_2^+ \epsilon_2 \operatorname{grad} \varphi_2\, dS\,;$$

$$W_2(\varphi_1, \varphi_2) = \frac{1}{2} \iiint_{V_2} [\text{grad } \varphi_2^+ \, \epsilon_2 \text{ grad } \varphi_2] \, dV +$$
$$\iint_{S_{21}} [(\Phi_{S21} - \varphi_2) \, n_2^+ \, \epsilon_2 \text{ grad } \varphi_2] \, dS + \tag{25}$$
$$+ \frac{1}{2} \iint_{S_3} [(\varphi_1 - \varphi_2) \, n_2^+ \, \epsilon_2 \text{ grad } \varphi_2 + \varphi_2 n_1^+ \, \epsilon_1 \text{ grad } \varphi_1) \, dS \, .$$

W_1 and W_2 are energies of subregions 1 and 2, respectively. This is understood for W_1 as follows. A double layer of moment $\epsilon_1(\Phi_{S11} - \varphi_1)$ is arranged on surface S_{11}, and that of moment $\epsilon_1(\varphi_2 - \varphi_1)$ on surface S_3. Beyond this, a surface charge of $n_2^+\epsilon_2$ *grad* φ_2 occurs on surface S_3. Functional W_1 contains surface energy, the energy of the supposed double layer and of the surface charge. Energy of subregion 2 can be interpreted similarly. Since in subregion 1 function φ_1 is the potential function and the solution should be satisfied in the case of various functions φ_2, therefore the necessary condition of a minimum of W_1 is that its first variation with respect to φ_1 should be zero, independent of φ_2. Similarly in subregion 2, the first variation of functional W_2 should be zero, independently of φ_1. Accordingly the necessary condition of the minimum is given by

$$\delta_1 W_1(\varphi_1, \varphi_2) = 0 \, , \tag{26}$$
$$\delta_2 W_2(\varphi_1, \varphi_2) = 0 \, , \tag{27}$$

where δ_1 denotes the first variation with respect to φ_1, and δ_2 that with respect to φ_2. Form these variations:

$$\delta W_1 = \iiint_{V_1} [\text{grad } \eta^+ \, \epsilon \text{ grad } \varphi_1] \, dV + \iint_{S_{11}} [-\eta n_1^+ \epsilon_1 \text{ grad } \varphi_1 +$$
$$+ (\Phi_{S11} - \varphi_1) \, n_1^+ \, \epsilon_1 \text{ grad } \eta) dS +$$
$$+ \frac{1}{2} \iint_{S_3} [-\eta n_1^+ \, \epsilon_1 \text{ grad } \varphi_1 + (\varphi_2 - \varphi_1) \, n_1^+ \, \epsilon_1 \text{ grad } \eta + \tag{28}$$
$$+ n_2^+ \, \epsilon_2 \text{ grad } \varphi_2) \, dS = 0 \, ,$$

$$\delta_2 W_2 = \iiint_{V_2} [\textit{grad } \xi^+ \, \epsilon_2 \text{ grad } \varphi_2] \, dV +$$
$$+ \iint_{S_{21}} [-\xi n_2^+ \, \epsilon_2 \text{ grad } \varphi_2] \, dS + \iint_{S_{21}} [(\Phi_{S21} - \varphi_2) \, n_2^+ \, \epsilon_2 \text{ grad } \xi] \, dS + \tag{29}$$
$$+ \frac{1}{2} \iint_{S_3} [-\xi n_2^+ \, \epsilon_2 \text{ grad } \varphi_2 \, (\varphi_1 - \varphi_2) \, n_2^+ \, \epsilon_2 \text{ grad } \xi +$$
$$+ \xi n^+ \, \epsilon_1 \text{ grad } \varphi_1] \, dS = 0 \, ,$$

where η and ξ are arbitrary, continuously derivable functions. Performing the suitable transformations and reductions, necessary conditions of the existence of the minimum are:

$$\begin{aligned}\delta_1 W_1 = & - \iiint_{V_1} [\eta \operatorname{div}(\epsilon_1 \operatorname{grad} \varphi_1)] \, dV + \\ & + \iint_{S_{11}} [(\Phi_{S11} - \varphi_1) n_1^+ \epsilon_1 \operatorname{grad} \eta] \, dS + \iint_{S_{12}} [\eta n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 \, dS] + \\ & + \frac{1}{2} \iint_{S_3} [(\varphi_2 - \varphi_1) n_1^+ \epsilon_1 \operatorname{grad} \eta + \\ & + \eta(n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 + n^+ \epsilon_2 \operatorname{grad} \varphi_2)] \, dS = 0 ,\end{aligned} \tag{30}$$

$$\begin{aligned}\delta_2 W_2 = & - \iiint_{V_2} [\xi \operatorname{div} (\epsilon_2 \operatorname{grad} \varphi_2)] \, dV + \\ & + \iiint_{S_{21}} (\Phi_{S21} - \varphi_2) n_2^+ \epsilon_2 \operatorname{grad} \xi \, dS + \iint_{S_{22}} [\xi n_2^+ \epsilon_2 \operatorname{grad} \varphi_2 \, dS + \\ & + \frac{1}{2} \iint_{S_3} [(\varphi_1 - \varphi_2) n_2^+ \epsilon_2 \operatorname{grad} \xi + \xi(n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 + \\ & + n_2^+ \epsilon_2 \operatorname{grad} \varphi_2)] \, dS = 0 .\end{aligned} \tag{31}$$

For arbitrary functions η and ξ, conditions (30) and (31) can be satisfied only for integrands of zero value, independently of η and ξ, that is, if conditions

$$\operatorname{div} (\epsilon_1 \operatorname{grad} \varphi_1) = 0 , \tag{32}$$

$$\Phi(S_{11}) = \varphi_1(S_1) , \tag{33}$$

$$n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 / S_{12} = 0 , \tag{34}$$

$$\varphi_1(S_3) = \varphi_2(S_3) , \tag{35}$$

$$n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 / S_3 + n_2^+ \epsilon_2 \operatorname{grad} \varphi_2 / S_3 = 0 , \tag{36}$$

are satisfied for all functions φ_2, and conditions

$$\operatorname{div} (\epsilon_2 \operatorname{grad} \varphi_2) = 0, \tag{37}$$

$$\Phi_{S21} = \varphi_2(S_1) \tag{38}$$

$$n_2^+ \epsilon_2 \operatorname{grad} \varphi_2 / S_2 = 0 , \tag{39}$$

$$\varphi_2(S_3) = \varphi_1(S_3) , \tag{40}$$

$$n_1^+ \epsilon_1 \operatorname{grad} \varphi_1 / S_3 + n_2^+ \epsilon_2 \operatorname{grad} \varphi_2 / S_3 = 0 , \tag{41}$$

are satisfied for all functions φ_1. These are identical with Eqs (16) to (26).

Accordingly, in the case of a dielectric medium continuous in subregions, potential function φ_1 valid for subregion 1 is obtained by minimizing functional W_1 with respect to φ_1, and potential function φ_2 valid for subregion 2 by minimizing functional W_2 with respect to φ_2. The two minimum conditions should be satisfied simultaneously.

Numerical approximations

The solution, that is, the potential function where functional $W(\varphi)$ is minimal, is determined by the numerical approximation method of Ritz and Galyerkin. The method is described in details in [2] and [3], therefore only the train of thought underlying the solution and the results are described here.

With the method Ritz and Galyerkin, the exact potential function is approximated by the linear combination of the first n elements of the complete system of functions, where n is a finite number:

$$\varphi \approx \varphi_n = \sum_{k=1}^{n} a_k f_k , \tag{42}$$

where f_k is the k-th element of the complete system of functions, a_k is the coefficient to be determined. Coefficients a_k to the required approximate potential function φ_n are determined from the condition of the minimum of functional $W(\varphi_n)$. Accordingly

$$\frac{\partial}{\partial a_k} W(a) = 0 \; ; \quad k = 1, 2, \ldots n , \tag{43}$$

where a is a column matrix of n elements consisting of the required coefficients. The way of writing the system of equations for determining the coefficients of the system of equation is discussed separately for the inhomogeneous dielectric medium and for a dielectric continuous by subregions.

a) *Inhomogeneous dielectric medium*

In the case of an inhomogeneous dielectric medium, functional (6) is to be minimized. The approximate function system is, according to (42):

$$\varphi_n = \sum_{k=1}^{n} a_k f_k . \tag{44}$$

From condition (43) for the minimum we obtain a linear system of equations for coefficients a_k, written in matrix form as

$$\mathbf{A}\, a = b\ , \tag{45}$$

where $\mathbf{A}$ is a quadratic matrix of n-th order and its j-th element in the i-th row being:

$$\begin{aligned} A_{i,j} = A_{j,i} = \iiint_V [\operatorname{grad} f_i^+ \,\epsilon\, \operatorname{grad} f_j]\, \mathrm{d}V - \\ - \iint_{S_1} [n^+ \,\epsilon\, \operatorname{grad}(f_i f_j)]\, \mathrm{d}S\,, \end{aligned} \tag{46}$$

b is a column matrix of n elements, its i-th element being:

$$b_i = - \iint_{S_1} [\Phi_{S1} n^+ \,\epsilon\, \operatorname{grad} f_i]\, \mathrm{d}S\,. \tag{47}$$

The required coefficients are

$$a = \mathbf{A}^{-1}\, b\ . \tag{48}$$

The approximate function system has to be chosen to that the zero of the function system is outside the examined region, with the exception of the case where this point coincides with a prescribed point of zero potential. A better approximation is obtained by transforming the function so that a point of the region with prescribed potential is zero and this point is exactly the zero of the function system. Accordingly, introduce the transformed potential function φ_n' defined as

$$\varphi_n' = \varphi_n - \Phi_0\,, \tag{49}$$

where

$$\Phi_0 = \Phi_{S1}(P)\,, \tag{50}$$

and $\Phi_{S1}(P)$ is the potential of an arbitrary point P of the surface of prescribed potential. Hence:

$$\varphi_n'(P) = 0. \tag{51}$$

In the course of solution, by substituting function $\varphi_n' + \Phi_0$ for φ_n in functional (6), function φ_n' is approximated according to (42):

$$\varphi_n' = \sum_{k=1}^{n} a_k' f_k'\,. \tag{52}$$

An approximating function system will be selected so that the zero is at P, thus potential function satisfies the boundary condition at one point. Determination of the coefficients for the transformed function system differs from the preceding only by the elements of b':

$$b'_i = - \iint_{S_1} [(\Phi_{S1} - \Phi_0) n^+ \epsilon \operatorname{grad} f'_i] \, dS . \tag{53}$$

The coefficients of the transformed function system are:

$$a' = \mathbf{A}^{-1} b' . \tag{54}$$

b) *Dielectric medium of continuous permittivity in subregions*

In the case of a dielectric medium of continuous permittivity in subregions, functional W_1 (24) and W_2 (25) are to be minimized with respect to φ_1, φ_2, respectively. Solution refers to the following transformed function:

$$\varphi'_{1n} = \varphi_{1n} - \Phi_{01} ; \quad \varphi'_{2n} = \varphi_{2n} - \Phi_{02} , \tag{55}$$

where Φ_{01} and Φ_{02} are the prescribed potentials of arbitrary points P_1, P_2 of surfaces S_{11} and S_{21}, respectively. The transformed functions are approximated by the function system:

$$\varphi'_{1n} = \sum_{k=1}^{n} a'_k f'_k ; \quad \varphi'_{2m} = \sum_{k=1}^{m} b'_k g'_k , \tag{56}$$

where f'_k and g'_k are the k-th elements of a complete function system ($f'_k = g'_k$ and $n = m$ being possible), further the zeros of φ'_{1n} and φ'_{2m} are at points P_1, and P_2 respectively. The necessary condition of the existence of a minimum is given by

$$\frac{\partial W_1}{\partial a'} = 0 ; \quad \frac{\partial W_2}{\partial b'} = 0 . \tag{57}$$

Performing the operations results a linear system of equations for coefficients a' and b', given in the matrix form as:

$$\mathbf{A}\, a + \mathbf{C}\, b = e , \tag{58}$$

$$\mathbf{D}\, a + \mathbf{B}\, b = h . \quad (59$$

The j-th element of the i-th row of quadratic matrices $\mathbf{A}$ and $\mathbf{B}$, of the n-th and m-th order are respectively:

$$\begin{aligned} A_{i,j} = A_{j,i} = & \iiint_{V_1} [\operatorname{grad} f'^{+}_i \epsilon_1 \operatorname{grad} f'_j] \, dV + \\ & + \iint_{S_{11}} [-n^+_1 \epsilon_1 \operatorname{grad} (f'_i f'_j)] \, dS + \frac{1}{2} \iint_{S_3} [-n^+_1 \epsilon_1 \operatorname{grad} (f'_i f'_j)] \, dS , \end{aligned} \tag{60}$$

and

$$B_{i,j} = B_{j,i} = \iiint_{V_2} [\operatorname{grad} g_i'^{+} \, \epsilon_2 \operatorname{grad} g_j'] \, \mathrm{d}V + \\ + \iint_{S_{21}} [-n_2^{+} \, \epsilon_2 \operatorname{grad} (g_i' g_j')] \, \mathrm{d}S + \frac{1}{2} \iint_{S_2} [-n_2^{+} \, \epsilon_2 \operatorname{grad} (g_i' g_j')] \, \mathrm{d}S \, . \tag{61}$$

The j-*th* element of the i-th row of matrix **C** of nxm order is:

$$C_{i,j} = \frac{1}{2} \iint_{S_2} [g_j' \, n_1^{+} \, \epsilon_1 \operatorname{grad} f_i' + f_i' \, n_2^{+} \, \epsilon_2 \operatorname{grad} g_j') \, \mathrm{d}S \, , \tag{62}$$

The j-th element of the i-th row of matrix **D** of mxn order is:

$$D_{i,j} = \frac{1}{2} \iint_{S_2} [f_j' \, n_2^{+} \, \epsilon_2 \operatorname{grad} g_i' + g_i' \, n_1^{+} \, \epsilon_1 \operatorname{grad} f_j'] \, \mathrm{d}S \, , \tag{63}$$

e is a column vector of n elements, where the i-th element is:

$$e_i = - \iint_{S_{11}} [(\Phi_{S11} - \Phi_{01}) \, n_1^{+} \, \epsilon_1 \operatorname{grad} f_i'] \, \mathrm{d}S - \\ - \frac{1}{2} \iint_{S_2} [(\Phi_{02} - \Phi_{01}) \, n_1^{+} \, \epsilon_1 \operatorname{grad} f_i' \, \mathrm{d}S \, , \tag{64}$$

h is a column vector of m elements, where the i-th element is:

$$h_i = - \iint_{S_{21}} [(\Phi_{S11} - \Phi_{02}) \, n_2^{+} \, \epsilon_2 \operatorname{grad} g_i'] \, \mathrm{d}S - \\ - \frac{1}{2} \iint_{S_2} [(\Phi_{01} - \Phi_{02}) \, n_2^{+} \, \epsilon_2 \operatorname{grad} g_i'] \, \mathrm{d}S \, . \tag{65}$$

The required coefficients are:

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} e \\ h \end{bmatrix} \tag{66}$$

Relationships necessary to determine the coefficients can be written in any system of co-ordinates. In the case of a planar problem, the relationships can be applied formally, taking into consideration the remarks made previously.

Accuracy of solution improves by increasing the number of elements in the approximate function system. On a digital computer, improved accuracy primarily means an increase in running time and only a lesser increase of storage capacity requirement.

Examples

In the following, application of the two methods will be demonstrated on the example of plane condenser with laminated dielectric medium.

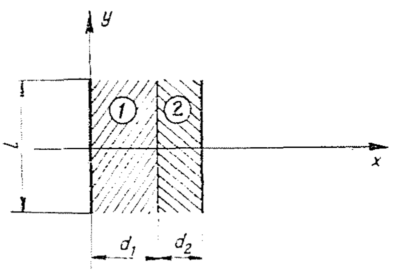

Fig. 3

Data of the laminated plane condenser shown in Fig. 3:

$$d_1 = 2 \text{ cm}; \quad \varepsilon_{1r} = 1; \quad d_2 = 1 \text{ cm}; \quad \varepsilon_{2r} = 2; \quad 1 = 1 \text{ cm}.$$

Voltage between the two electrodes $U_0 = 1$ V. Calculations involve:

$$\frac{\partial}{\partial y} = 0 .$$

a) First the dielectric medium is considered to be inhomogeneous. Approximate the potential function by two terms. Zero the potential of the electrode in plane $x = 0$, thus:

$$\varphi_2 = a_1 f_1 + a_2 f_2$$

where

$$f_1 = x; \quad f_2 = x^2.$$

Relationships (46) and (47) are applied for a two-dimensional problem, considering that tensor ϵ is diagonal, a discontinuous function. Taking these into consideration, the value of e. g. A_{11} is calculated as follows:

$$A_{11} = \int\limits_{x=0}^{d_1} \int\limits_{y=-1/2}^{1/2} \left[\frac{\partial f_1}{\partial x}\varepsilon_1 \frac{\partial f_1}{\partial x}\right] dy\, dx + \int\limits_{x=d_1}^{d_1+d_2} \int\limits_{y=-1/2}^{1/2} \left[\frac{\partial f_1}{\partial x}\varepsilon_2 \frac{\partial f_1}{\partial x}\right] dy\, dx -$$

$$- \int\limits_{-1/2}^{1/2} \left[-\varepsilon_1 \frac{\partial}{\partial x}(f_1^2)\Big|_{x=0}\right] dy - \int\limits_{-1/2}^{1/2} \left[\varepsilon_2 \frac{\partial}{\partial x}(f_1^2)\Big|_{x=d_1+d_2}\right] dy ,$$

where ε_1 and ε_2 denote the scalar permittivities of the layers. Performing the operations:

$$A_{11} = \varepsilon_0[2 + 2 + 0 - 2 \cdot 2 \cdot 3] = -8\varepsilon_0 .$$

Omitting further calculations we obtain:

$$\mathbf{A} = -\begin{bmatrix} 8 & 40 \\ 40 & 1{,}5467.10^2 \end{bmatrix};$$

$$b = -\begin{bmatrix} 2 \\ 12 \end{bmatrix};$$

The coefficients are:

$$a = \begin{bmatrix} 4{,}7058 \cdot 10^{-1} \\ -4{,}4120 \cdot 10^{-1} \end{bmatrix};$$

Table 1 contains exact and approximate potential values φ_p and φ_k, respectively further percentage errors for various x values. Fig. 4 shows functions φ_p and φ_k vs. x.

$\frac{x}{\text{cm}}$	$\frac{\varphi_p}{V}$	$\frac{\varphi_k}{V}$	$h = \frac{\varphi_k - \varphi_p}{\varphi_p} \cdot 100$
0.0	0.00	0.00	0.00
0.2	0.08	0.09	12.50
0.4	0.16	0.18	12.50
0.6	0.24	0.27	12.50
0.8	0.32	0.35	9.38
1.0	0.40	0.43	7.50
1.2	0.48	0.50	4.17
1.4	0.56	0.57	1.79
1.6	0.64	0.64	0.00
1.8	0.72	0.70	−2.78
2.0	0.80	0.76	−5.00
2.2	0.84	0.82	−2.38
2.4	0.88	0.88	−0.00
2.6	0.92	0.93	1.09
2.8	0.96	0.97	1.04
3.0	1.00	1.01	1.00

Table 1.

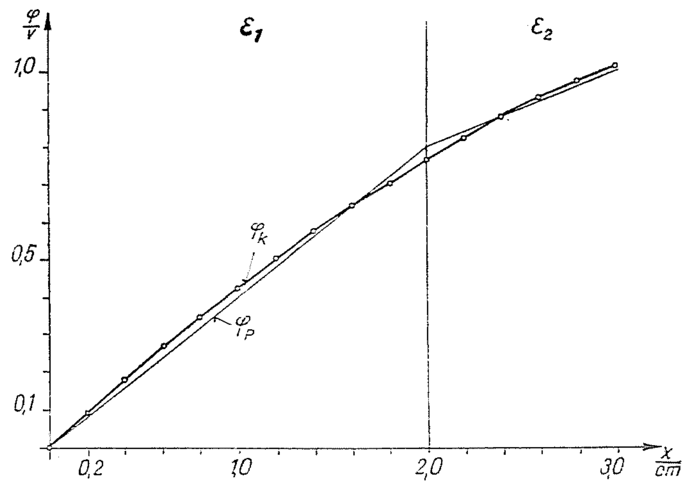

Fig. 4

b) Hereafter the dielectric medium is considered to be homogeneous by subregions. For the transformation choose the values:

$$\Phi_{01} = 0 \text{ and } \Phi_{02} = 1 \ .$$

Hence, approximate functions φ'_{1n} and φ'_{2n} are zero at $x = 0$ and $x = d_1 + d_2$. Approximate both functions by one term. In this case

$$\varphi'_1 = a_1 f'_1; \quad \varphi'_2 = b_1 g'_1$$

where

$$f'_1 = x \ ; \quad g'_1 = (x - d_1 - d_2) \ .$$

Relationships (60) to (65), yield coefficients a_1, b_1. Expressing e.g. A_{11}.

$$A_{11} = d_1\varepsilon_1 - 2\ \varepsilon_1 \cdot 0 - \frac{1}{2}\varepsilon_1 \cdot 2\ d_1 = 0\ .$$

Other coefficients are calculated similarly, hence:

$$\begin{bmatrix} a_1 \\ \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}(d_1\varepsilon_2 + d_2\varepsilon_1) \\ \\ -\frac{1}{2}(d_1\varepsilon_2 + d_2\varepsilon_1) & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2}\varepsilon_1 \\ \\ -\frac{1}{2}\varepsilon_2 \end{bmatrix};$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \frac{1}{d_1\varepsilon_2 + d_2\varepsilon_1} \begin{bmatrix} \varepsilon_2 \\ \varepsilon_1 \end{bmatrix}$$

The potential functions are:

$$\varphi_1 = a_1 f'_i = \frac{\varepsilon_2}{d_1\varepsilon_2 + d_2\varepsilon_1}\, x, \ \textit{if}\ 0 \leq x \leq d_1$$

$$\varphi_2 = b_1 g'_1 + \Phi_{02} = \frac{\varepsilon_1}{d_1\varepsilon_2 + d_2\varepsilon_1}\,(x - d_1\, d_2) + 1, \text{ if } d_1 \leq x \leq d_1 + d_2\,,$$

equal to the exact potential given by the elementary method.

Approximation has led to the exact solution, due to the approximation of a linear function by a linear function and applying function transformation. This simple problem evidences the improvement of the solution upon taking the break in the potential function into consideration, instead of considering the break in the permittivity function as a general inhomogeneity.

Summary

Calculation of the static and stationary electromagnetic field in linear, inhomogeneous, anisotropic media is discussed in the case of mixed-type boundary conditions. By applying variational calculation, both Maxwell equations and boundary conditions are satisfied at the zero variation of the suitable functional written for the potential function. The solution is the stationary function of such a functional, to be determined by the Ritz—Galyerkin numerical approximation method. The way of taking into consideration the break of the derivatives of the potential function, in the case of a medium continuous in subregions is described. Two numerical examples are presented as an illustration.

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István BÁRDI,
Oszkár BÍRÓ, H-1521 Budapest
Miklós GYIMESI