A VARIATIONAL METHOD FOR SOLVING HEAT CONDUCTIONAL PROBLEMS*

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1. Outlines of the method

Let A denote the set of functions y

i) defined on the finite closed interval $[a, b]$;

ii) piecewise continuously differentiable there;

iii) satisfying the boundary conditions

$$y(a) = y_a; \quad y(b) = y_b. \tag{1}$$

Let the functional $F\{y\}$ defined for all $y \in A$ take its unique relative minimum at $y_0 \in A$. The function y_0 minimizing the functional F is sought for. The numerical calculation of y_0 may be carried out in the following manner.

Let us divide the interval $[a, b]$ into n subintervals ($n > 1$). The points of subdivision are $x_0 = a, x_1, \ldots, x_n = b$. A trial function $y_{(0)}$ is taken arbitrarily: it will be the zeroth approximation. To the kth approximation $y_{(k)}(x)$ a function y^* is constructed:

$$y^*(x) = y_{(k)}(x) + \Delta_i(x), \tag{2}$$

where

$$\Delta_i(x) = \begin{cases} \dfrac{\Delta}{x_i - x_{i-1}}(x - x_{i-1}), & \text{if} \quad x \in [x_{i-1}, x_i], \\ \dfrac{\Delta}{x_i - x_{i+1}}(x - x_{i+1}), & \text{if} \quad x \in [x_i, x_{i+1}], \\ 0 & \end{cases} \tag{3}$$

otherwise.

Here Δ is an arbitrary (but fixed) value, and $i \in \{1, 2, \ldots, n-1\}$ That is the function $y_{(k)}(x)$ is "varied" in the neighborhood of point x_i (Lagrange's method of the calculus of variations [1]).

Next the values $F\{y_{(k)}\}$ and $F\{y^*\}$ are compared. If $F\{y^*\} < F\{y_{(k)}\}$ then y^* is a "better" approximation than $y_{(k)}$; therefore the procedure is continued using y^* instead of $y_{(k)}$. For $F\{y^*\} \geq F\{y_{(k)}\}$ the values i or/and Δ are changed and another y^* is sought for.

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Conveniently, a linear function may be used for $y_{(0)}$. In this case, the approximations produced by the above procedure are polygonal functions, and elements of set A. Hence, this method is essentially a combination of the Lagrange method and the Euler method [1] of the calculus of variations.

Making full use of the change of Δ and i, the "best" approximation of y_0 with respect to the functional F is found among the polygonal functions in A which consist of n linear parts.

2. A heat conduction problem

Let us consider a rod along the axis x with the terminals $x = 0$ and $x = L$. Let the heat conductivity λ depend on the space co-ordinate x and the temperature $T : \lambda = \lambda(x, T)$. Assume λ to be piecewise continuously differentiable with respect to both its variables. The temperatures of the terminals are kept constant in time:

$$T(0) = T_1; \quad T(L) = T_2. \tag{4}$$

Now, our aim is the solution of the one-dimensional steady-state equation of heat conduction:

$$\frac{d}{dx}\left(\lambda(x, T)\frac{dT}{dx}\right) = 0 \tag{5}$$

with boundary conditions (4). In general, the solution cannot be given in closed form, so approximate calculations are justified.

Our method will be illustrated on the variational problem [2]

$$F\{T\} \equiv \int_0^L \frac{\lambda(x, T)}{2}\left(\frac{dT}{dx}\right)^2 dx - \frac{(T_2 - T_1)^2}{2\int_0^L \frac{1}{\lambda(x, T)}\, dx} = \text{minimum} \tag{6}$$

taken from Gyarmati's Governing Principle of Dissipative Processes [3]. The functional $F\{T\}$ in (6) is seen to satisfy the requirements in item 1. furthermore, the desired stationary temperature distribution T_0 to be the unique solution of the extremum problem (6). Hence the method outlined in 1. can be applied for the numerical computation of T_0.

3. Numerical calculations

The method in 1. is suitable for computer processing. It is easy to program and a significant advantage is to require a small storage capacity: in case of n points of subdivision only $n + 2$ or $2n + 4$ storage units are needed (depending upon the applied program), while n^2 would be necessary in case of the more usual method (reducing the problem to solving a system of linear algebraic equations). Our method suits even a small programmable calculator in case of 8 to 10 points of subdivision.

As an illustrative example let us consider a rod of length 10 units. Let the boundary conditions (4) now be

$$T(0) = 0; \quad T(10) = 1000. \tag{7}$$

Let the heat conductivity depend on x and T as follows:

$$\lambda = 1 + \frac{x}{10} + \frac{T}{1000}. \tag{8}$$

The temperature values at integer values of $x(T(1), T(2), \ldots, T(9))$ are wanted.

Starting from the linear temperature distribution as a trial function, compute the values $(\Delta F)_i = F\{T + \Delta_i\} - F\{T\}$ belonging to variations $\Delta = 0.1$ and $\Delta = -0.1$, respectively, for $i = 1, 2, \ldots, 9$. Only the negative ΔF will be used for each i. A new trial function is constructed as follows: increase (or decrease) the value of the trial function by 1 in the point where $|\Delta F|$ is maximum and by a less amount, proportional to the local value of $|\Delta F|$, in the other points. Of course, the change follows the negative direction of ΔF. If the function T^{**} constructed in such a way meets $F\{T^{**}\} < F\{T\}$, then the procedure is repeated using T^{**} as a new trial function. Otherwise the original trial function is the first approximation.

The second approximation is obtained after decreasing both the variation Δ and the change value by one order of magnitude (to 0.01 and 0.1, respectively) and using the first approximation as trial function. Approximations of higher order are obtained in the same manner.

Table 1 shows the numerical data. The exact values have been obtained from the analytic solution of the problem. Approximations closer than the fourth one are seen to be possible only by increasing the number of the points of subdivision.

Table 1

	Trial function	1st approx.	2nd approx.	3rd approx.	4th approx.	5th approx.	Exact solution
T(0)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
T(1)	100.0	131.92	165.25	168.43	168.73	168.73	168.94
T(2)	200.0	247.10	301.07	306.19	306.66	306.67	306.94
T(3)	300.0	353.51	418.42	424.55	425.12	425.13	425.40
T(4)	400.0	455.56	522.93	529.28	529.88	529.88	530.13
T(5)	500.0	552.62	617.57	623.74	624.32	624.33	624.55
T(6)	600.0	650.72	705.05	710.18	710.66	710.67	710.85
T(7)	700.0	738.68	785.49	789.99	790.42	790.43	790.56
T(8)	800.0	834.58	861.89	864.47	864.71	864.71	864.81
T(9)	900.0	914.93	932.48	934.19	934.35	934.36	934.40
T(10)	1000	1000	1000	1000	1000	1000	1000

Summary

A combination of the Lagrange method and the Euler method of the calculus of variations has been developed for computer processing. As an example, a nonlinear differential equation has been considered: the equation of steady-state heat conduction in an inhomogeneous rod of temperature-dependent conductivity.

References

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