REDUCTION OF THE DEGREES OF FREEDOM OF LINEAR-ELASTIC MECHANICAL MODELS

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1. Introduction

Structural vibration problems in mechanical engineering, vehicle industry, lightweight building methods, acoustics require knowledge of vibration patterns and natural frequencies as a rule. In case of certain structures, these magnitudes are easy to determine by analytic means. In other cases, mathematical formulation of the problem may lead to formulae containing independent variables by the hundreds or thousands. Handling of problems of this size encounters difficulties even with the latest computer technique. In certain cases it may be advisable to reduce the number of variables by some or other method under the common name of reduction of the degrees of freedom [1].

Without entering in details of mathematical simulation, replacing the models of divided parameters (e.g. plates, bars) by those of discrete parameters means *a priori* a reduction of the degree of freedoms by substituting finite for infinite degrees of freedom. Reduction of the degrees of freedom in course of simulation will be ignored.

2. Models of finite degrees of freedom

Let the undamped, linear-elastic model containing elements of concentrated parameters have n degrees of freedom. In case of free vibration, motion equations lead to the well-known eigenvalue problem of the form

$$\mathbb{C}s = \omega^2 \mathbb{M}s \tag{1}$$

where the column matrix s of n elements includes amplitudes s_i $(i, \ldots, n)$ of the displacement co-ordinates, quadratic matrices $\mathbb{M}$ and $\mathbb{C}$ the characteristics of inertia and of elasticity, resp., and ω is the circular frequency of vibration.

There are several mathematical ways to achieve the goal set in (1) i.e. reduction of the extension of the eigenvalue problem. This goal may be justified from several aspects.

In applying the method of finite elements, often the volume of the problem is prohibitive, n being several hundreds or thousands. Limitations of the computer technique impose to eliminate some variables, involving mathematical reduction of the degrees of freedom.

In the following, two possibilities of reducing the degrees of freedom will be presented. The first one results in a reduced-order problem formally similar to the linear eigenvalue problem in (1); the second leads to a nonlinear eigenvalue problem, terms "linear" and "non-linear" in the respective headings refer to the outcome of reduction.

2.1 Linear reduction of the degrees of freedom

This means to reduce the degrees of freedom is essentially an approximation, featured by assigning to the original model of degrees of freedom reduced by pure mathematical means, thus it mathematically reduces the degrees of freedom of the model also in the physical meaning by eliminating displacements irrelevant to the kinetic energy.

Let the displacements to be eliminated concentrated in vector v of m elements. After convenient row and column exchanges, Eq.(1) may be transcribed into:

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \omega^2 \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2}$$

or, in particular:

$$\mathbf{C}_{11}u + \mathbf{C}_{12}v = \omega^2\mathbf{M}_{11}u + \omega^2\mathbf{M}_{12}v, \tag{3a}$$

$$\mathbf{C}_{21}u + \mathbf{C}_{22}v = \omega^2\mathbf{M}_{21}u + \omega^2\mathbf{M}_{22}v. \tag{3b}$$

From (3b):

$$v = (\mathbf{E} - \omega^2\mathbf{C}_{22}^{-1}\mathbf{M}_{22})^{-1}\mathbf{C}_{22}^{-1}(\omega^2\mathbf{M}_{21} - \mathbf{C}_{21})u, \tag{4a}$$

or simplified:

$$v = \mathbf{Q}(\omega)u, \tag{4b}$$

where $\mathbf{E}$ is a unit matrix of order m.

At this point, the theory may have different issues. Simply resubstituting (4a) into (3a), reducing and factorizing yields:

$$\mathbf{S}(\omega)u = 0 \tag{5}$$

where elements of the coefficient matrix $\mathbf{S}$ are not linear functions of the circular frequency ω of vibration any more. This case will be expounded under 2.2.

A linear expression results e.g. from applying the approximation

$$Q(\omega)\approx Q(\bar{\omega})$$

in Eq. (4b), where $\bar{\omega}$ is a fix value chosen "near" the wanted circular frequency of vibration. Thereby, after resubstitution into (3a), a linear eigenvalue problem remains to be solved. An iteration is started thereby, the determined ω value being given to $\bar{\omega}$, the procedure will be iterated [9]. The convergence depends, however, in addition to the given problem, also on the initial $\bar{\omega}$ value, besides, in numerical analyses, also inversions in (4a) may raise difficulties.

Further methods have been published by e.g. HURTY [2], CRAIG and BAMPTON [3], GERADIN [4], with the common feature to approximate from above the lower circular frequencies of the original problem by the circular natural frequencies of the reduced problem. The most effective method in this category has been suggested by ANDERSON, IRONS and ZIENKIEWICZ [5].

According to their method, if convergency exists, the first factor in the right-hand side of (4a) may be written as:

$$(\mathbf{E}-\omega^2\mathbf{C}_{22}^{-1}\mathbf{M}_{22})^{-1}=\mathbf{E}+\omega^2\mathbf{C}_{22}^{-1}\mathbf{M}_{22}+(\omega^2\mathbf{C}_{22}^{-1}\mathbf{M}_{22})^2+\dots \qquad (6)$$

Using (4a) and (6), Eq.(3a) becomes

$$\bar{\mathbf{C}}u=\omega^2\bar{\mathbf{M}}u+\quad \text{higher-power terms in } \omega \qquad (7)$$

where

$$\bar{\mathbf{C}}=\mathbf{C}_{11}-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} \quad \text{and}$$

$$\bar{\mathbf{M}}=\mathbf{M}_{11}+\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{M}_{22}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{M}_{21}-\mathbf{M}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$$

are reduced spring and mass matrices.

Omitting terms of higher power in ω from (7) yields a linear approximation of the original problem (1):

$$\bar{\mathbf{C}}u=\bar{\omega}^2\bar{\mathbf{M}}u\,, \qquad (8)$$

$\bar{\omega}^2$ being the eigenvalue of the reduced problem. (8) is advantageous by involving the solution of an ordinary two-matrix eigenvalue problem, with library programs available.

Eigenvalues $\bar{\omega}_i$ of the reduced problem (8) can be demonstrated [6] to yield upper bounds for eigenvalues ω_i of the original problem (1) until right-hand side of (6) converges. This is valid for

$$\omega_i<\mu_1\,, \qquad (9)$$

where μ_1 is the first (least) eigenvalue of the so-called inner eigenvalue problem

$$\mathbf{C}_{22}v = \mu^2 \mathbf{M}_{22}v . \tag{10}$$

Furthermore, if eigenvalues of (10) are

$$\mu_1^2 < \mu_2^2 < \ldots < \mu_m^2 ,$$

in this order, then

$$\omega_1^2 \leqq \mu_1^2 \leqq \mu_m^2 \leqq \omega_n^2 ,$$

where ω_1^2 and ω_n^2 are least and highest eigenvalues, resp., of the original problem (1). Thus, the spectrum of eigenvalues of the inner problem is sited inside that of the original one. In practically existing structures, the limiting case $\omega_1 = \mu_1$ is exceptional, permitting to assume $\mu_1 > \omega_1$. The convergency, hence the approximation from above of a given ω_i is the better, the less is ω_i with respect to μ_1, i.e., the higher is μ_1. Thus, displacements to be eliminated are advisably chosen from the parts of the structure with the least dynamic elasticity.

2.2 Nonlinear reduction of the degrees of freedom

This way of reduction lessens the number of degrees only in the mathematical sense, hence the number of explicit degrees of freedom in the equations is less than n. In this method, no new model of physically reduced degrees of freedom will be assigned the original model like in the former item. The present method will yield a nonlinear eigenvalue problem. Its solution is rather time consuming but its accuracy can be increased in principle to infinity, contrary to the former one. An undesirable by-effect, however, to be set out later, does intervene. For the sake of a detailed illustration, let us start from the former relationship (5), where $\mathbb{S}$ is the reduced dynamic stiffness matrix of the system. The order n decreased to $n-m$.

Remind that elements of $\mathbb{S}$ are no more linear functions of the circular natural frequencies ω^2 of the system, but quotients of the polynomials of ω^2. For $n, m \to \infty$, in some cases the elements of $\mathbb{S}$ may be expressed in terms of trigonometric and hyperbolic functions of ω^2 [7]. Because of the inversion in (4a), $\mathbb{S}$ can only be produced from numerical ω values. At the same time it is a restriction of the applicable mathematical algorithms.

(5) may be fulfilled either as

$$\det \mathbb{S} = 0 , \tag{11a}$$

or as

$$u = 0 . \tag{11b}$$

In the reduced problem, also (11b) may be meaningful, namely (5) contains only $n-m$ degrees of freedom of the original system, thus, validity of (11b) does not exclude vibration. These natural frequencies—if any—and the pertaining vibration patterns can be calculated using Eq. (3). Thus, knowledge of the complete dynamic stiffness matrix of the system is needed. From Eq. (3b) it is obvious that in case of $u=0$—excluding the trivial case $v=0$—$\mathbf{S}$ is not interpreted, the frequency function det $\mathbf{S}$ is singular at this place. Thus, natural circular frequencies meeting Eq. (11b) cannot be calculated exclusively from (11a).

Often nothing but the reduced problem can be written, hence only (11a) is available as a frequency function. The reduction of freedom degrees according to this item has a special consequence: not all natural frequencies can always be calculated from the frequency equation (11a) but "holes" are left in the set of frequencies. As seen by the simple numerical example in [1], this phenomenon occurs if the system has a natural circular frequency that is at the same time circular frequency of (a) subsystem(s) assumed to join its surrounding at fixed points.

In case of a discrete model, it is relatively simple to decide whether natural frequencies have been omitted in the calculation or not. The number of natural frequencies—including also the multiplicities,—has to equal the degrees of freedom of the original problem. In case of continua, it is a rather intricate problem.

3. Continuum models

A theoretical possibility of the vibration analysis of composite systems (see e.g. items 3 and 4 in [8]) is by decomposing the structure of infinite degrees of freedom *via* the assumption of nodes into subsystems of known field functions for free or excited vibrations. Points of the subsystem joining the structure nodes are termed pole points. Assuming a vibration of a circular frequency ω, field functions permit to produce for each subsystem the displacement vectors s_i of pole points and the force effect vectors q_i acting at the pole points—considered as outer ones from the aspect of the subsystem. These vectorial equations are the parametric equation system of the vibration, where the parameter is a vector containing arbitrary constants of the field functions. Eliminating the parameters from the equations, a direct relationship results between the external force effects acting at the pole points and displacements. This step is equivalent to the nonlinear reduction of freedom degrees described under 2.2, namely the degrees of freedom to be applied thereafter in the equations equal the sum of the finite degrees of freedom of the pole points. Accordingly:

$$\mathbf{S}_i s_i = q_i ,$$

where $\mathbf{S}_i$ is the reduced dynamic stiffness matrix of the i-th subsystem. Notice that relying on direct physical considerations, immediately $\mathbf{S}_i$ will be written, hence directly the reduced dynamic stiffness matrix will be started from.

In this method the reduced dynamic stiffness matrix $\mathbf{S}$ of the entire structure will be built up from the $\mathbf{S}_i$, thus, any element of the set of frequencies is left out from the calculation that is a natural frequency of a subsystem vibrating with rigidly clamped poles.

To decide whether a natural frequency below some frequency has been omitted or not, WITTRICK and WILLIAMS [7] suggested a computation algorithm.

To avoid these shortcomings, it is advisable to choose a method without reduction of the degrees of freedom, theoretically excluding the possibility of omitting a natural frequency (e.g. item 5 in [8]).

Summary

In dynamically analysing linear-elastic structures, the degrees of freedom of the mathematical model can be reduced by purely mathematical means.

The reduction of the degrees of freedom will be separately considered for models of finite and infinite degrees of freedom. Pros and cons of the different methods will be examined. Finally, an adverse consequence of reduction in case of continuum models will be pointed out.

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