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RESEARCH ARTICLE

Maximum deflection of symmetric wall-frame buildings

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Abstract

The system of governing differential equations of lateral deflection of symmetric multi-storey buildings subjected to uniformly distributed horizontal load is presented. It is shown that the "standard" equivalent column approach (when the stiffnesses of the bracing units are added up) is only applicable to the deflection analysis in the rare case when the system only consists of shear walls and a single framework. When the bracing system contains more frameworks, then a more sophisticated approach is needed where the full interaction between the vertical elements in bending and shear may need to be taken into account.

Two new methods are developed for the determination of the maximum deflection of mixed bracing systems consisting of frameworks and shear walls: one is very simple while the other one is more accurate. The accuracy of both procedures is demonstrated using the results of over 200 bracing systems. The error range of the more accurate method is -4% to +4% when the buildings contain frameworks and shear walls/cores. A worked example and step-by-step instructions are presented to aid practical application.

Keywords

deflection · continuum method · multi-storey buildings · horizontal load

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1 Introduction

The deflection analysis of multi-storey frameworks has a long history. The mathematical problem of a cantilever composed of a number of parallel beams interconnected by cross bars (i.e., frameworks, using today's terminology) was presented and solved as early as in 1947 in a brilliant paper [Chitty, 1947]. Chitty and Wan [1948] then applied the method to tall buildings under wind load. However, the applicability of the original method was considerably restricted as they neglected the effect of the axial deformation of the columns. Numerous methods were then published, amazingly unaware of Chitty's efforts, both for individual frameworks or coupled shear walls [Csonka, 1950; Beck, 1956; Ligeti, 1974; Szmodits, 1975; Szerémi, 1984] and also for wall-frame buildings [Rosman, 1960; MacLeod, 1971; Despeyroux, 1972; Council, 1978; Stafford Smith *et al.*, 1981; Goschy, 1981; Hoenderkamp and Stafford Smith, 1984; Taranath, 1988; Coull, 1990; Schueller, 1990; Coull and Wahab, 1993]. The most comprehensive treatment, perhaps, is to be found in the excellent textbook by Stafford Smith and Coull [1991] where a whole chapter is devoted to individual frameworks and another chapter deals with symmetric wall-frame buildings. Most of the methods, however, are too complicated, even as approximate methods, or neglect one or more significant phenomena in order to be able to offer relatively simple solutions. Furthermore, none of them are backed up with a comprehensive accuracy analysis and, as a result, their applicability is not possible to establish for practical structural engineering problems. Some are based on the equivalent column approach and use a procedure whereas the characteristic stiffnesses are simply added up for the analysis. This approach – although perfectly legitimate for stability and frequency analyses – is not acceptable for the deflection analysis, as it will be demonstrated in this paper. All the above shortcomings were addressed in a recent paper [Zalka, 2009] which offered a closed-form solution for the deflection of symmetric buildings. However, that solution is still fairly complicated and, as it will be shown later on, its accuracy can significantly be improved.

The aim of this paper is twofold:

- to present two new approximate procedures which can be used in practice for the determination of the maximum deflection of symmetric multi-storey buildings
- to demonstrate the accuracy of the two procedures, based on the results of over 200 test cases

The continuum method will be used and it will be assumed for the analysis that the structures are

- regular in the sense that their characteristics do not vary over the height
- at least four storeys high with identical storey heights
- sway structures with built-in lower end at ground floor level and free upper end

and that

- the floor slabs have great in-plane and small out-of-plane stiffness
- the deformations are small and the material of the structures is linearly elastic

2 The governing differential equations of lateral deflection of symmetric wall-frame buildings

Symmetric cross wall-frame buildings under horizontal load develop lateral deflection in the direction of the external load. As the resultant of the horizontal load passes through the shear centre of the bracing system (O), no torsion occurs. A typical building is shown in Figure 1. The bracing system of such buildings may consist of frameworks, coupled shear walls, shear walls and cores. Coupled shear walls can be considered frameworks if the width of the wall sections and the shear deformation of the connecting beams are taken into account. From now on, frameworks also represent coupled shear walls.

The building then can be modelled by a planar system of the bracing units which are linked by incompressible pinned bars representing the floor slabs. Figure 2 shows a typical model where the first f bracing units may represent frameworks and coupled shear walls and the remaining m bracing units may be shear walls and cores.

When the lateral load of a multi-storey building is resisted by this system of f frameworks and m shear walls/cores, the behaviour of the system is complex. As a rule, the frameworks develop a deflection shape which is a combination of bending and shear deformation. The deflection shape of the shear walls and cores is of "pure" bending. The floor slabs of the building, being stiff in their plane, make the bracing units assume the same deflection shape. As the two types would have different shapes on their own, they interact and this interaction results in the "compromise" deflection of the system.

The characteristics of the interaction can be best investigated by using the governing differential equations of the system and

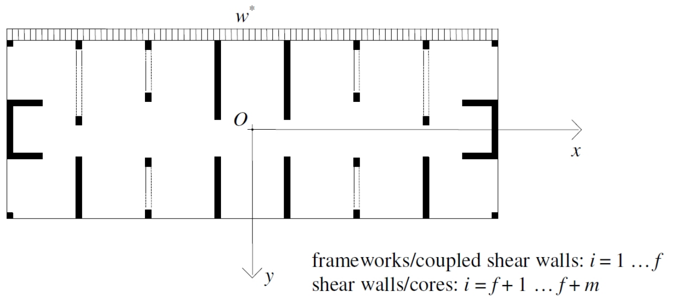

Fig. 1. Symmetric cross wall–frame building with f frameworks/coupled shear walls and m shear walls/cores.

analysing the different roles that the two different bracing types play. The system of governing differential equations of f frameworks and m shear walls/cores consists of two sets of equations. The first set represents f compatibility conditions for the f frameworks expressing continuity at the vertical lines of contraflexure of the beams of the frameworks. (The frameworks at this stage are single-bay structures but the final results will be valid for multi-bay frameworks as well.)

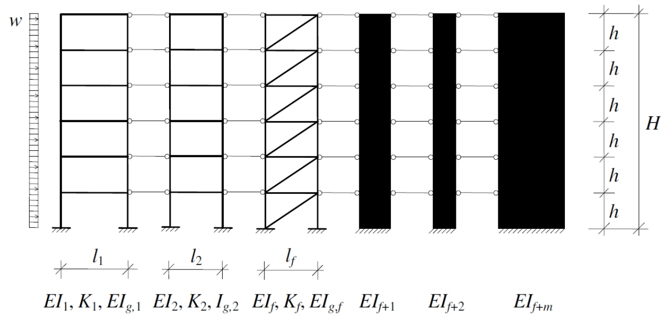

Fig. 2. A planar system of f frameworks and m shear walls/cores.

Based on the derivation regarding a single framework under uniformly distributed horizontal load [Zalka, 2009], these equations are as follows:

$$y_1'' - \frac{l_1}{K_1}N_1'' + \frac{l_1}{EI_{g,1}}N_1 = 0 \tag{1}$$

$$y_2'' - \frac{l_2}{K_2}N_2'' + \frac{l_2}{EI_{g,2}}N_2 = 0 \tag{2}$$

$$y_f'' - \frac{l_f}{K_f}N_f'' + \frac{l_f}{EI_{g,f}}N_f = 0 \tag{3}$$

In the above f equations $EI_{g,i}$ is the global bending stiffness of the ith framework (with $i = 1\ldots f$) – see Equation (27) for the determination of $I_{g,i}$. Term K_i represents the shear stiffness:

$$K_i = \left(\frac{1}{K_{c,i}} + \frac{1}{K_{b,i}}\right)^{-1} = K_{b,i}\frac{K_{c,i}}{K_{b,i}+K_{c,i}} = K_{b,i}r_i \tag{4}$$

The shear stiffness has two "components"; $K_{b,i}$ is related to the beams while $K_{c,i}$ is linked to the columns of the framework. They are defined as

$$K_{b,i} = \frac{12EI_{b,i}}{l_i h} \quad \text{and} \quad K_{c,i} = \frac{12EI_{c,i}}{h^2} \tag{5}$$

where $I_{b,i}$ and $I_{c,i}$ are the sums of the second moments of area of the beams and columns, respectively, of the ith framework.

A second set of equations is needed as in the above equations, in addition to the deflection (y_i), the normal forces that originate from the bending of the beams of the frameworks (N_i) are also unknown quantities. This second set (of $f + m$ equations) represents the bending of the vertical elements, i.e., the full-height columns of f frameworks and m shear walls/cores:

$$y_1'' EI_1 = -M_1 + l_1 N_1 \tag{6}$$

$$y_2'' EI_2 = -M_2 + l_2 N_2 \tag{7}$$

$$y_f'' EI_f = -M_f + l_f N_f \tag{8}$$

and

$$y_{f+1}'' EI_{f+1} = -M_{f+1} \tag{9}$$

$$y_{f+2}'' EI_{f+2} = -M_{f+2} \tag{10}$$

$$y_{f+m}'' EI_{f+m} = -M_{f+m} \tag{11}$$

where the first shear wall/core is marked by subscript $f + 1$. Bending stiffness EI_i for the frameworks ($1 \le i \le f$) is determined using the sum of the second moments of area of the columns ($I_{c,i}$), adjusted by parameter r_i [Equation (4)], resulting in the local bending stiffness of the ith framework as

$$EI_i = EI_{c,i} r_i \tag{12}$$

The bending stiffness of the shear walls/cores ($f + 1 \le i \le m$) is determined in the usual manner.

Moment M_i in the above equations is the moment share on the ith bracing unit, according to

$$M_i = q_i M \tag{13}$$

where

$$M = \frac{wz^2}{2} \tag{14}$$

is the total external moment on the system and q_i is the apportioner of the external load. Its value is determined according to the "overall stiffness" of the bracing unit in question:

$$q_i = \frac{S_i}{\sum\limits_{i=1}^{f+m} S_i} \tag{15}$$

The "overall stiffness" of a bracing unit (either a framework or a shear wall/core) is defined as

$$S_i = \frac{1}{y_i(H)} \tag{16}$$

where $y_i(H)$ is the maximum deflection of the ith unit. It follows from the above equations that the relationship

$$w_i = q_i w \tag{17}$$

also holds, expressing the load share on the ith bracing unit.

The above two sets of differential equations represent the complete governing differential equations of the bracing system consisting of frameworks and shear walls/cores. The first set consists of f equations and these equations are responsible for fulfilling the compatibility conditions. The second set consists of $f + m$ equations in two parts. The first part (with f equations) represent the bending of the columns of the frameworks and the second part (with m equations) stand for the bending of the shear walls/cores.

There are two possibilities to proceed from here. One approach leads to a very simple solution and the other approach results in a more accurate solution. Both solutions are important. Although the more accurate solution will be recommended for use regarding this planar problem, the simple solution will play an important role when the torsional behaviour of asymmetric buildings are investigated (in a follow-up paper).

3 A simple solution

A close look at the two sets of equations reveals the fact that the second part of the second set [Equations (9),(10) and (11)] are not directly needed for the solution. The solution of the problem requires $2f$ equations and Equations (1), (2), (3) and (6), (7), (8) represent $2f$ equations. Setting Equations (9), (10) and (11) aside is equivalent to taking the shear walls/cores out of the system and creating two sub-systems: the frameworks and the shear walls/cores. Naturally, both sub-systems have their own external load share. The load that belongs to the frameworks is defined by apportioners $q_1, q_2, \ldots q_f$ and the load on the shear walls/cores is determined by $q_{f+1}, q_{f+2}, \ldots q_{f+m}$.

Consider first the first sub-system of f frameworks. Equations (1), (2) and (3) represent the compatibility conditions of the f frameworks and Equations (6), (7) and (8) stand for the bending of the vertical elements of system, i.e., the full-height bending of the columns. The normal forces from the compatibility equations can be eliminated using the relevant equations in the second set.

In doing so, the governing equation of the ith framework of the first sub-system (with $1 \le i \le f$) is obtained as

$$y_i'' - \frac{1}{K_i}\left(y_i'' EI_i + M_i\right)'' + \frac{1}{EI_{g,i}}\left(y_i'' EI_i + M_i\right) = 0 \tag{18}$$

Introducing Equations (13) and (14) and after some rearrangement, Equation (18) can be written as

$$y_i'''' - y_i''\left(\frac{K_i}{EI_i} + \frac{K_i}{EI_{g,i}}\right) = \frac{q_i w}{EI_i}\left(\frac{z^2}{2}\frac{K_i}{EI_{g,i}} - 1\right) \tag{19}$$

The structure of Equation (19) clearly shows that, as a rule, it is not possible to create an equivalent column in such a way that the corresponding stiffnesses of the frameworks (EI_i, $EI_{g,i}$ and K_i) are simply added up. This is a significant observation as the situation with the stability and frequency analyses is completely different: the solution of the frequency and stability problems is based on an equivalent column whose characteristic stiffnesses are obtained by adding up the stiffnesses of the individual bracing units [Zalka, 2013].

The governing differential equations of the second sub-system of m shear walls/cores (with $f+1 \le i \le m$) can be expressed in a similar (but much simpler) form:

$$y_i'' EI_i = -\frac{q_i w z^2}{2} \tag{20}$$

Using the above consecutive $f+m$ equations would lead to the *complete* system of governing differential equations of the whole system. However, there is no need for this procedure that would lead to a fairly complicated solution. Two important observations can be made that make it possible to simplify the deflection problem:

a) According to the assumption regarding the floor slabs, all the bracing units assume the same deflection shape, i.e., $y_1 = y_2 = \ldots = y_i = y$.

b) The fact that the bracing units take on the external load according to their stiffness [Equations (16) and (17)] makes it possible to concentrate on one bracing unit only.

If a framework is to be used for the determination of the deflection of the building, then the solution of Equation (19) is needed. The short form of Equation (19) is

$$y_i'''' - \kappa_i^2 y_i'' = \frac{q_i w}{EI_i}\left(\frac{a_i z^2}{2} - 1\right) \tag{21}$$

where

$$\kappa_i = \sqrt{a_i + b_i}, \quad a_i = \frac{K_i}{EI_{g,i}}, \quad b_i = \frac{K_i}{EI_i} \tag{22}$$

The structure of the above differential equation is identical to that of a single, *independent* framework and therefore its solution can be directly applied. Bearing in mind that the deflection of the ith framework is identical to the deflection of the whole system, the formula for the deflection of the system is obtained as

$$\begin{aligned} y(z) = y_i(z) = {} & \frac{q_i w}{EI_{f,i}}\left(\frac{H^3 z}{6} - \frac{z^4}{24}\right) + \frac{q_i w z^2}{2K_i s_i^2} \\ & - \frac{q_i w EI_i}{K_i^2 s_i^3}\left(\frac{\cosh \kappa_i (H-z) + \kappa_i H \sinh \kappa_i z}{\cosh \kappa_i H} - 1\right) \end{aligned} \tag{23}$$

where

$$EI_{f,i} = E(I_i + I_{g,i}) \tag{24}$$

is the sum of the local and global bending stiffnesses and

$$s_i = 1 + \frac{a_i}{b_i} = \frac{I_{g,i} + I_i}{I_{g,i}} = 1 + \frac{I_i}{I_{g,i}} \tag{25}$$

Maximum deflection develops at $z = H$:

$$y_{\max} = y_{i,\max} = \frac{q_i w H^4}{8EI_{f,i}} + \frac{q_i w H^2}{2K_i s_i^2} - \frac{q_i w EI_i}{K_i^2 s_i^3}\left(\frac{1 + \kappa_i H \sinh \kappa_i H}{\cosh \kappa_i H} - 1\right) \tag{26}$$

Although the original derivations assume single-bay frameworks, the formulae for the deflection (given here and also in Section 4) are also applicable to multi-bay frameworks if the basic stiffness characteristics (I_i, $I_{g,i}$ and K_i) are calculated in such a way that the number of bays is taken into account. This leads to simple summations for I_i and K_i. As for $I_{g,i}$, the second moments of area of the cross-sections of all the columns should be taken with regard to the centroid of the cross-sections:

$$I_{g,i} = \sum_{j=1}^{n} A_j t_j^2 \tag{27}$$

where A_j is the cross-sectional area of the jth column of the ith framework, t_j is its distance from the centroid of the cross-sections of the columns and n is the number of (full-height) columns.

If a shear wall is to be used for the determination of the deflection of the building, then the solution of Equation (20) is needed:

$$y(z) = y_i(z) = \frac{q_i w}{EI_i}\left(\frac{H^3 z}{6} - \frac{z^4}{24}\right) \tag{28}$$

and the maximum deflection is

$$y_{\max} = y_{i,\max} = \frac{q_i w H^4}{8EI_i} \tag{29}$$

The beauty of this solution is in its simplicity. It should be noted, however, that the determination of the load share on the bracing unit that is used for the calculation of the deflection of the building requires the determination of the maximum deflection of *every* bracing unit of the bracing system – see Equations (15) and (16). Equations (26) and (29) can be used for this purpose. An arbitrary apportioner, say $q_i = 1$, can be used for these calculations as the intensity of the load drops out of the formulae.

The drawback of this procedure lies in the fact that in the process of separating the two sub-systems the direct interaction between the shear walls and the frameworks is tacitly ignored. This fact – and the numerical consequences regarding accuracy – are spectacularly shown in Figure 6. A comprehensive accuracy analysis is presented in Section 5.

4 A more accurate solution

The accuracy of the procedure presented in the previous section can be improved if the direct interaction between the shear walls and frameworks is taken into account.

Before this step is taken, it is worth analysing the structures of the governing differential equations. It is also useful to consider the different nature of the interaction among the individual frameworks, the individual shear walls, and between the frameworks and the shear walls. When frameworks of different stiffnesses are considered, there is an interaction because (due to the different stiffnesses) their deflections are of different shape. (The only exception is when the frameworks are identical.) When shear walls are considered, there is no interaction because their deflection *shapes* are identical. When a system of frameworks and shear walls is considered, there is always an interaction because their deflection shapes are always different:

the frameworks develop a mixture of bending and shear deformation while the shear walls are always in pure bending.

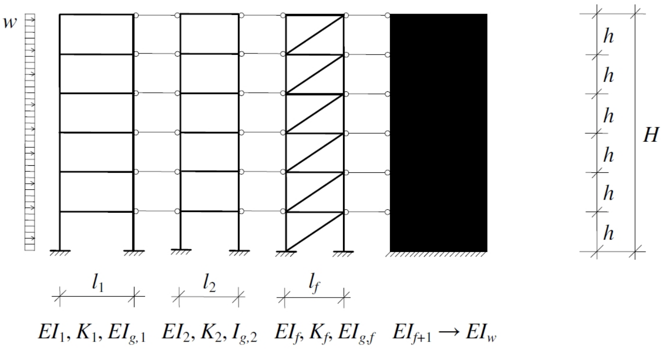

Fig. 3. A system of f frameworks and one shear wall.

It should be noted that the equations in the second part of the second set [i.e., Equations (9), (10) and (11)] are of the same structure and, more importantly, they do not contain normal forces N_i that are needed for the overall solution. The mathematical consequence of this is that these equations are not needed directly for the solution of the deflection problem from the point view of the frameworks. (This fact was utilized in the previous section when the two groups – frameworks and shear walls – were effectively separated.) The practical consequence of this is that any number of shear walls can be "put together" (by adding up their bending stiffnesses) for the deflection analysis. This also follows from the fact that there is no interaction among the shear walls themselves, whose deflection *shapes* (in pure bending) are of the same nature and their load is proportional to their stiffnesses.

The problem of f frameworks and m shear walls is thus reduced to a system of f frameworks and one shear wall, accompanied by differential equations (1), (2), (3), (6), (7), (8) and (9). For practical reasons, subscript $f+1$ is replaced with the more meaningful w as it refers to the shear wall (Figure 3).

Instead of separating the different types of bracing unit (and losing the effect of direct interaction), the shear wall will now be incorporated into the system of frameworks. The investigation of a single framework and one shear wall (Figure 4) shows how this can be achieved.

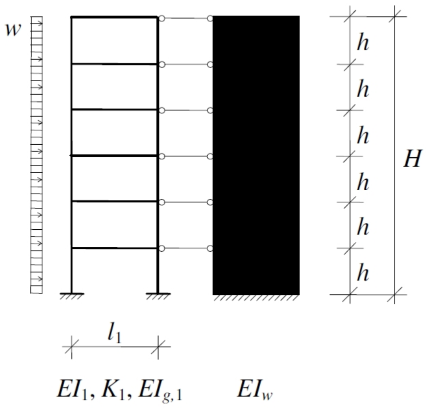

Fig. 4. A system of a single framework and one shear wall.

The differential equations of this system are Equations (1), (6) and (9). With $y_1 = y_2 = y$, and using subscript w referring to the shear wall, they assume the form

$$y'' - \frac{l_1}{K_1} N_1'' + \frac{l_1}{EI_{g,1}} N_1 = 0 \tag{30}$$

$$y'' EI_1 = -M_1 + l_1 N_1 \tag{31}$$

$$y'' EI_w = -M_w \tag{32}$$

where M_1 and M_w are the moment shares on the framework and the shear wall, respectively. It is clear that Equations (31) and (32) can be combined: they represent the same type of bending (i.e., pure bending), their left-hand side only contain bending stiffness and they stand for the same deflection shape y:

$$y'' E(I_1 + I_w) = -M + l_1 N_1 \tag{33}$$

with $M = M_1 + M_w$ being the total external moment.

Altogether, two equations are needed for the final solution (y and N_1 being the two unknowns) and Equations (30) and (33) furnish these two equations. In practical terms, it can be said that the shear wall has been "pushed" into the framework, increasing its local bending stiffness. There is another important aspect of this procedure. By incorporating the shear wall into the framework, the interaction between the framework and the shear wall is automatically taken into account through the solution of Equations (30) and (33) as I_w is now part of the system to be solved. This is what we have referred to in the beginning of this section as "direct interaction". (In the previous section when we presented the "simple solution", the second moments of area of the walls were not part of the system to be solved as the shear walls were separated into another sub-system.)

The above equations also demonstrate the precise meaning of the term "wall-frame interaction". The term is normally interpreted as the interaction between the two bracing units, i.e., the shear wall and the framework. It may be more to the point to refer to this phenomenon as the interaction between the bending and shear deformations.

The situation is similar, although slightly more complicated when the system consists of f frameworks and one shear wall (that, as we saw above, may be the sum of several shear walls). The number of equations needed for the solution is $2f$. The choice for one set of f equations is obvious: the compatibility equations represented by Equations (1), (2) and (3). The question arises, how to obtain the second set of f equations. Proceeding as with the case of the single frame–single wall above, the differential equation of the shear wall [Equation (32)] should be combined with those representing the bending of the vertical elements of the frameworks [Equations (6), (7) and (8)]. This task seems to be difficult – if not impossible – as there is only one shear wall and there are f frameworks, and f equations are needed. However, understanding the behaviour of the system during interaction points at the solution. Due to the floor slabs, the shear wall interacts with all the frameworks during deflection as, as a rule, their individual deflection shapes are different. It follows that all the frameworks participate when – as with the single frame–single wall case – the bending stiffness of the shear wall is added to the frame system. The "intensity" of the interaction depends on the stiffnesses of the participants. It follows that the bending stiffness of the shear wall should be apportioned among the frameworks according to their relative stiffnesses. This is achieved if apportioners $\bar{q}_i$ are used, that are only related to the first f bracing units, i.e., to the original frameworks:

$$\bar{q}_i = \frac{S_i}{\sum_{i=1}^{f} S_i} \tag{34}$$

The system of f frameworks and m shear walls has now been reduced to a system of f frameworks. However, these are not the original frameworks as the local bending stiffness of each framework is now amended by its share of the bending stiffness of the shear wall. Accordingly, the second set of equations assume the form

$$y_1'' E(I_1 + \bar{q}_1 I_w) = -M_1^* + l_1 N_1 \tag{35}$$

$$y_2'' E(I_2 + \bar{q}_2 I_w) = -M_2^* + l_2 N_2 \tag{36}$$

$$y_f'' E(I_f + \bar{q}_f I_w) = -M_f^* + l_f N_f \tag{37}$$

Including the stiffness of the shear wall(s) in the above equations also means that the interaction between the shear wall(s) and the frameworks is directly taken into account.

It should be noted that M_1^*, M_2^* and M_f^* in the above equations are different from their equivalents in Equations (6), (7) and (8) as the frameworks themselves are different from the original frameworks. Their value

$$M_i^* = q_i^* M \tag{38}$$

is determined using the new apportioner

$$q_i^* = \frac{S_i^*}{\sum_{i=1}^{f} S_i^*} \tag{39}$$

whose values are determined using the "new" frameworks. The "overall stiffness" of the ith "new" framework is defined as

$$S_i^* = \frac{1}{y_i^*(H)} \tag{40}$$

where $y_i^*(H)$ is the maximum deflection of the ith (new) framework. The load share on this framework is now

$$w_i^* = q_i^* w \tag{41}$$

The star in the above equations indicates that the frameworks in question differ from the original ones in that they also contain a portion of the bending stiffness of the shear wall.

It is now feasible to combine the two sets of differential equations: Equations (1), (2) and (3) representing the compatibility conditions of the f frameworks, and Equations (35), (36) and (37) representing the bending of the vertical elements of the bracing system *including* the shear walls incorporated into the frameworks. In doing so, the governing equation of the ith framework of the system is obtained as

$$y_i'' - \frac{1}{K_i}\left(y_i'' E I_i^* + M_i^*\right)'' + \frac{1}{EI_{g,i}}\left(y_i'' E I_i^* + M_i^*\right) = 0 \tag{42}$$

where

$$I_i^* = I_i + \bar{q}_i I_w \tag{43}$$

In the above equations K_i, EI_i and $EI_{g,i}$ are the stiffnesses of the ith (original) framework and EI_w is the bending stiffness of the shear wall.

Equation (42) is clearly analogous with Equation (18) and therefore the procedure presented in Section 3 can be repeated. This leads to the governing differential equation

$$y_i'''' - \kappa_i^{*2} y_i'' = \frac{q_i^* w}{EI_i^*}\left(\frac{a_i z^2}{2} - 1\right) \tag{44}$$

where

$$\kappa_i^* = \sqrt{a_i + b_i^*}, \quad a_i = \frac{K_i}{EI_{g,i}}, \quad b_i^* = \frac{K_i}{EI_i^*} \tag{45}$$

The solution – after amending the relevant bending stiffnesses – can also be used. The formulae for the deflection of the system is obtained as

$$\begin{aligned} y(z) = y_i^*(z) = {} & \frac{q_i^* w}{EI_{f,i}^*}\left(\frac{H^3 z}{6} - \frac{z^4}{24}\right) + \frac{q_i^* w z^2}{2 K_i s_i^{*2}} \\ & - \frac{q_i^* w E I_i^*}{K_i^2 s_i^{*3}}\left(\frac{\cosh \kappa_i^*(H - z) + \kappa_i^* H \sinh \kappa_i^* z}{\cosh \kappa_i^* H} - 1\right) \end{aligned} \tag{46}$$

where

$$EI_{f,i}^* = E(I_i^* + I_{g,i}) \tag{47}$$

is the sum of the local and global bending stiffnesses and

$$s_i^* = 1 + \frac{a_i}{b_i^*} = \frac{I_{g,i} + I_i^*}{I_{g,i}} = 1 + \frac{I_i^*}{I_{g,i}} \tag{48}$$

Maximum deflection develops at $z = H$:

$$\begin{aligned} y_{\max} = y_i^*(H) = {} & \frac{q_i^* w H^4}{8 EI_{f,i}^*} + \frac{q_i^* w H^2}{2 K_i s_i^{*2}} \\ & - \frac{q_i^* w E I_i^*}{K_i^2 s_i^{*3}}\left(\frac{1 + \kappa_i^* H \sinh \kappa_i^* H}{\cosh \kappa_i^* H} - 1\right) \end{aligned} \tag{49}$$

The situation is similar to that with the "simple solution" in Section 3 in that the determination of the load share on the framework ($q_i^* w$) that is used for the calculation of the deflection of the building requires the determination of the maximum deflection of *each* framework [cf. Equations (39) and (40)]. These values are calculated using Equation (49) with an arbitrary apportioner, say, $q_i^* = 1$, as the intensity of the load drops out of the formulae.

Again, the above equations spectacularly demonstrate that, as a rule, it is not possible to carry out the lateral deflection analysis of a building by adding up the corresponding stiffnesses of the bracing units in order to create an equivalent column, as is widely circulated in the literature. The equivalent column approach does work for the frequency and stability analyses [Zalka, 2013] but not for the deflection analysis. There is only one exception: a system of shear walls and a single framework.

5 Practical application: worked example

When the formulae for the maximum deflection were derived above, the presentation followed an order that was most suitable for, and in line with, the theoretical considerations. For practical applications, however, it is advisable to follow a different order to simplify and minimize the amount of calculation. This is shown here using a 28-storey building whose layout is shown in Figure 5. The building is subjected to a uniformly distributed horizontal load of intensity $w^* = 1\,\text{kN/m}^2$ in direction y.

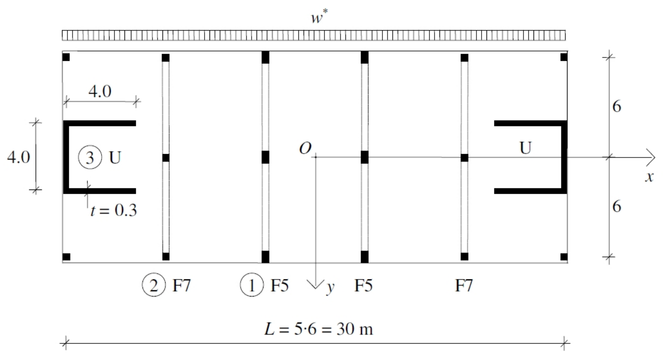

Fig. 5. Layout for the worked example.

The maximum deflection of the building will be determined using both methods. The building has a symmetric bracing system that consists of four frameworks and two cores. Because of symmetry, it is possible to consider half of the system (with half of the external load: $w = w^*L/2 = 15\,\text{kN/m}$). The storey-height is $h = 3\,\text{m}$ and the total height of the building is $H = 28 \times 3 = 84\,\text{m}$. The modulus of elasticity is $E = 25 \times 10^6\,\text{kN/m}^2$. The cross-sectional characteristics of the frameworks are given in Table 1. The relevant second moment of area of the core (I_x) is $I_w = 11.245\,\text{m}^4$.

The Finite Element based computer program Axis [2003] gives $y = 0.1844\,\text{m}$ as the maximum deflection of the building. This value is considered the "exact" solution.

Tab. 1. Cross-sectional characteristics for frameworks F5 and F7.

Bracing unit	cross-section of columns [m]	cross-section of beams [m]	$I_{c,i}$ [m^4]	$I_{b,i}$ [m^4]	$I_{g,i}$ [m^4]
1: F5	0.4×0.7	0.4×0.4	0.0343	$0.0042\dot{6}$	20.16
2: F7	0.4×0.4	0.4×0.4	0.0064	$0.0042\dot{6}$	11.52

5.1 Solution 1: A simple solution

The calculation is best carried out in two steps:

1) The basic stiffness characteristics, the maximum deflection, the overall stiffness and the apportioner for each bracing unit are calculated ($EI, EI_g, K, y_{\max}, S, q$)

2) The maximum deflection of the building is determined using any of the bracing units [Equation (26) or Equation (29)]

1) The basic characteristics for each bracing unit

Framework F5 With the part shear stiffnesses given by Equations (5)

$$K_{b,1} = \frac{12EI_b}{lh} = \frac{12 \cdot 25 \cdot 10^6 \cdot 0.0042\dot{6}}{6 \cdot 3} = 71111\,\text{kN},$$

$$K_{c,1} = \frac{12 \cdot 25 \cdot 10^6 \cdot 0.0343}{3^2} = 1143333\,\text{kN}$$

the shear stiffness of the framework is calculated using Equation (4)

$$K_1 = K_{b,1}\frac{K_{c,1}}{K_{b,1}+K_{c,1}} = K_{b,1}r_1 = 71111\frac{1143333}{71111+1143333}$$
$$= 71111 \cdot 0.9414 = 66947\,\text{kN}$$

which also furnishes the value of parameter $r_1 = 0.9414$.

The local bending stiffness is given by Equation (12):

$$EI_1 = EI_{c,1}r_1 = 25 \cdot 10^6 \cdot 0.0343 \cdot 0.9414 = 807250\,\text{kNm}^2$$

The global bending stiffness is calculated using Equation (27):

$$EI_{g,1} = E\sum_{j=1}^{n} A_j t_j^2 = 25 \cdot 10^6 \cdot 0.4 \cdot 0.7 \cdot 6^2 \cdot 2$$
$$= 504000000\,\text{kNm}^2$$

The sum of the local and global stiffnesses [Equation (24)] is:

$$EI_{f,1} = EI_1 + EI_{g,1} = 504807250\,\text{kNm}^2$$

With auxiliary quantities a_1, b_1, s_1 and κ_1 obtained from Equations (22) and (25) as

$$a_1 = \frac{K_1}{EI_{g,1}} = \frac{66947}{504000000} = 0.000133,$$
$$b_1 = \frac{K_1}{EI_1} = \frac{66947}{807250} = 0.08293,$$
$$s_1 = 1 + \frac{a_1}{b_1} = 1 + \frac{0.000133}{0.08293} = 1.0016,$$
$$\kappa_1 = \sqrt{a_1 + b_1} = \sqrt{0.000133 + 0.08293} = 0.288,$$
$$\kappa_1 H = 24.2$$

the maximum deflection of the framework is calculated using Equation (26) (with $q_1 = 1$):

$$y_1 = \frac{15 \cdot 84^4}{8 \cdot 504807250} + \frac{15 \cdot 84^2}{2 \cdot 66947 \cdot 1.0016^2}$$
$$- \frac{15 \cdot 807250}{66947^2 \cdot 1.0016^3}\left(\frac{1 + 24.2\sinh 24.2}{\cosh 24.2} - 1\right)$$
$$= 0.185 + 0.788 - 0.063 = 0.910\,\text{m}$$

The overall stiffness of the framework is given by Equation (16):

$$S_1 = \frac{1}{y_1(H)} = \frac{1}{0.91} = 1.10\,\text{m}^{-1}$$

Framework F7 A copycat calculation leads to the overall stiffness of the framework.

With the part shear stiffnesses given by Equations (5)

$$K_{b,2} = \frac{12EI_b}{lh} = \frac{12 \cdot 25 \cdot 10^6 \cdot 0.0042\dot{6}}{6 \cdot 3} = 71111\,\text{kN},$$

$$K_{c,2} = \frac{12 \cdot 25 \cdot 10^6 \cdot 0.0064}{3^2} = 213333\,\text{kN}$$

the shear stiffness of the framework is calculated using Equation (4)

$$K_2 = K_{b,2}\frac{K_{c,2}}{K_{b,2} + K_{c,2}} = K_{b,2}r_2 = 71111\frac{213333}{71111 + 213333} = 71111 \cdot 0.75 = 53333\,\text{kN}$$

which also furnishes the value of parameter $r_2 = 0.75$.

The local bending stiffness is given by Equation (12):

$$EI_2 = EI_{c,2}r_2 = 25 \cdot 10^6 \cdot 0.0064 \cdot 0.75 = 120000\,\text{kNm}^2$$

The global bending stiffness is calculated using Equation (27):

$$EI_{g,2} = E\sum_{j=1}^{n} A_j t_j^2 = 25 \cdot 10^6 \cdot 0.4 \cdot 0.4 \cdot 6^2 \cdot 2 = 288000000\,\text{kNm}^2$$

The sum of the local and global stiffnesses [Equation (24)] is

$$EI_{f,2} = EI_2 + EI_{g,2} = 288120000\,\text{kNm}^2$$

With auxiliary quantities a_2, b_2, s_2 and κ_2 obtained from Equations (22) and (25) as

$$a_2 = \frac{K_2}{EI_{g,2}} = \frac{53333}{288000000} = 0.000185,$$

$$b_2 = \frac{K_2}{EI_2} = \frac{53333}{120000} = 0.44444,$$

$$s_2 = 1 + \frac{a_2}{b_2} = 1 + \frac{0.000185}{0.44444} = 1.000416,$$

$$\kappa_2 = \sqrt{0.000185 + 0.44444} = 0.6668,$$

$$\kappa_2 H = 56.0$$

the maximum deflection of the framework is calculated using Equation (26) (with $q_2 = 1$):

$$y_2 = \frac{15 \cdot 84^4}{8 \cdot 288120000} + \frac{15 \cdot 84^2}{2 \cdot 53333 \cdot 1.000416^2} - \frac{15 \cdot 120000}{53333^2 \cdot 1.00416^3}\left(\frac{1 + 56\sinh 56}{\cosh 56} - 1\right) = 0.324 + 0.991 - 0.035 = 1.28\,\text{m}$$

The overall stiffness of the framework is given by Equation (16):

$$S_2 = \frac{1}{y_2(H)} = \frac{1}{1.28} = 0.78\,\text{m}^{-1}$$

U-core The maximum deflection of the core is calculated using Equation (29) (with $q_3 = 1$):

$$y_3 = \frac{wH^4}{8EI_w} = \frac{15 \cdot 84^4}{8 \cdot 25 \cdot 10^6 \cdot 11.245} = 0.332\,\text{m}$$

and the stiffness of the core is

$$S_3 = \frac{1}{y_3(H)} = \frac{1}{0.332} = 3.01\,\text{m}^{-1}$$

The three apportioners are determined using Equation (15):

$$q_1 = \frac{S_1}{\sum_{i=1}^{f+m} S_i} = \frac{1.1}{1.1 + 0.78 + 3.01} = 0.225,$$

$$q_2 = \frac{0.78}{4.89} = 0.16,$$

$$q_3 = \frac{3.01}{4.89} = 0.615$$

2) The maximum deflection of the building

The maximum deflection of the building is calculated using the U-core with its load share [Equation (29)]:

$$y_{\max} = y_3(H) = \frac{q_3 wH^4}{8EI_3} = \frac{0.615 \cdot 15 \cdot 84^4}{8 \cdot 25 \cdot 10^6 \cdot 11.245} = 0.204\,\text{m}$$

This value is 10.6% greater than the “exact” (computer based) solution. Naturally, the same value is obtained using the two frameworks with their load shares.

5.2 Solution 2: A more accurate solution

The procedure for the more accurate solution can be organized into three steps.

1) The basic stiffness characteristics, the maximum deflection, the overall stiffness and the apportioner for each framework are calculated (EI, EI_g, K, $y_{\max}$, S, $\overline{q}$)

2) Using apportioners $\overline{q}$, the bending stiffness of each framework is amended ($EI \to EI^*$). All characteristics that are affected are re-calculated for each framework (y^*, S^*, q^*)

3) The maximum deflection of the building is determined using any of the frameworks [Equation (49)]

1) The basic characteristics for each framework

This task has already been completed in Section 5.1 and the results will be used below.

2) New bending stiffness and new characteristics for the frameworks

Framework F5* According to Equation (43), a portion of the second moment of area of the shear wall that is proportional to the overall stiffness of framework F5 is added to its original second moment of area. The apportioner is given by Equation (34). The amended local bending stiffness is

$$EI_1^* = E(I_1 + \overline{q}_1 I_w) = 25 \cdot 10^6\left(0.0343 \cdot 0.9414 + \frac{1.1}{1.1 + 0.78}11.245\right) = 165295282\,\text{kNm}^2$$

Because of this change, three other parameters have to be amended, according to Equations (45) and (48):

$$b_1^* = \frac{K_1}{EI_1^*} = \frac{66947}{165295282} = 0.000405,$$

$$s_1^* = 1 + \frac{a_1}{b_1^*} = 1 + \frac{0.000133}{0.000405} = 1.328$$

$$\kappa_1^* = \sqrt{a_1 + b_1^*} = \sqrt{0.000133 + 0.000405} = 0.0232 \quad \text{and}$$

$$\kappa_1^* H = 1.948$$

The sum of the local and global stiffnesses [Equation (47)] is

$$EI_{f,1}^* = EI_1^* + EI_{g,1} = 165295282 + 504000000$$
$$= 669295282\,\text{kNm}^2$$

Equation (49) (with $q_1^* = 1$) gives the maximum deflection of framework F5*:

$$y_1^* = \frac{15 \cdot 84^4}{8 \cdot 669295282} + \frac{15 \cdot 84^2}{2 \cdot 66947 \cdot 1.328^2} - \frac{15 \cdot 165295282}{66947^2 \cdot 1.328^3}\left(\frac{1 + 1.948 \sinh 1.948}{\cosh 1.948} - 1\right) = 0.316\,\text{m}$$

Its stiffness [Equation (40)] is

$$S_1^* = \frac{1}{y_1^*} = \frac{1}{0.316} = 3.164\,\text{m}^{-1}$$

Framework F7*   The procedure for the other framework is the same. Its amended local bending stiffness is

$$EI_2^* = E(I_2 + \bar{q}_2 I_w) = 25 \cdot 10^6 (0.0064 \cdot 0.75 + \frac{0.78}{1.1 + 0.78} 11.245)$$
$$= 116756968\,\text{kNm}^2$$

Because of this change, three other parameters have to be amended, according to Equations (45) and (48):

$$b_2^* = \frac{K_2}{EI_2^*} = \frac{53333}{116756968} = 0.000457,$$

$$s_2^* = 1 + \frac{a_2}{b_2^*} = 1 + \frac{0.000185}{0.000457} = 1.405,$$

$$\kappa_2^* = \sqrt{a_2 + b_2^*} = \sqrt{0.000185 + 0.000457} = 0.0253 \quad \text{and}$$

$$\kappa_2^* H = 2.128$$

The sum of the local and global stiffnesses [Equation (47)] is

$$EI_{f,2}^* = EI_2^* + EI_{g,2} = 116756968 + 288000000$$
$$= 404756968\,\text{kNm}^2$$

Equation (49) (with $q_2^* = 1$) gives the maximum deflection of framework F7*:

$$y_2^* = \frac{15 \cdot 84^4}{8 \cdot 404756968} + \frac{15 \cdot 84^2}{2 \cdot 53333 \cdot 1.405^2} - \frac{15 \cdot 116756968}{53333^2 \cdot 1.405^3}\left(\frac{1 + 2.128 \sinh 2.128}{\cosh 2.128} - 1\right) = 0.443\,\text{m}$$

Its stiffness [Equation (40)] is

$$S_2^* = \frac{1}{y_2^*} = \frac{1}{0.443} = 2.257\,\text{m}^{-1}$$

Equation (39) gives the new apportioners for the two frameworks:

$$q_1^* = \frac{S_1^*}{\sum_{i=1}^{f} S_i^*} = \frac{3.164}{3.164 + 2.257} = 0.584,$$

$$q_2^* = \frac{S_2^*}{\sum_{i=1}^{f} S_i^*} = \frac{2.257}{3.164 + 2.257} = 0.416$$

3) The maximum deflection of the two frameworks

These have already been calculated under a horizontal load of $w = 15\,\text{kN/m}$. According to Equation (49), the same calculation – but with the real load share of the framework – gives the maximum deflection of the building. Using framework F5*, this is

$$y_{\max} = q_1^* y_1^*(H) = 0.584 \cdot 0.316 = 0.184\,\text{m}$$

This value is practically identical with the "exact" (computer based) solution. Naturally, using the other framework with its load share leads to the same value.

The performance of the two approximate procedures presented in this paper and that of the "old" method [Zalka, 2009] is shown in Figure 6 where the height of the building varies between four and eighty storeys. The error is defined as the difference between the approximate and "exact" solutions, related to the "exact" solution. Positive errors indicate greater deflections, i.e., an approximation on the safe side.

The weakness of the simple method is spectacularly shown in Figure 6: it neglects the effect of the direct interaction between the shear walls and the frameworks. As a rule, this effect is smaller for very low and tall structures and greater for medium-rise buildings.

5.3 Practical considerations

In many practical cases a deflection analysis is needed in order to demonstrate that the maximum deflection of the structure does not exceed a certain value, say $H/500$, and the procedure is used as a checking mechanism. In such cases it is worth considering the use of one of the procedures in a simplified manner.

Equations (26) and (49) consist of three terms: the first two terms represent bending and shear deflections, respectively, while the third term is responsible for the interaction. It is perfectly clear from the equations that the effect of interaction is always beneficial. Neglecting the third term, therefore, represents an approximation on the safe side, while makes the calculation extremely simple – a true back-of-the-envelope procedure. If the building still meets the requirement regarding the maximum deflection, then it is not necessary to use the full formulae (with the hyperbolic terms that are not suitable for hand calculation).

6 Accuracy analysis

The results of the worked example (Figure 6) offer some indication regarding the accuracy of the two procedures ("simple

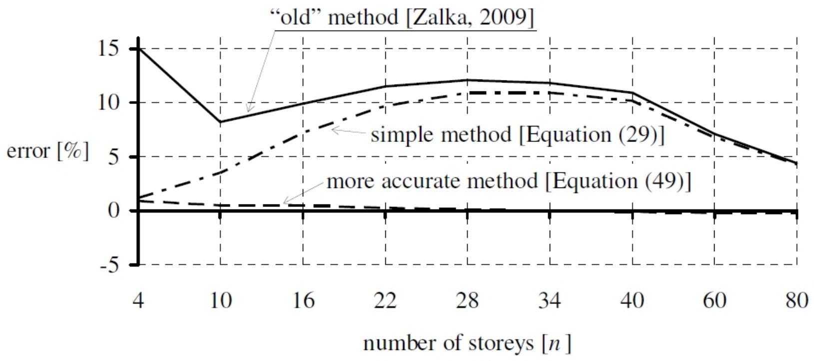

Fig. 6. Accuracy of the approximate methods over the height.

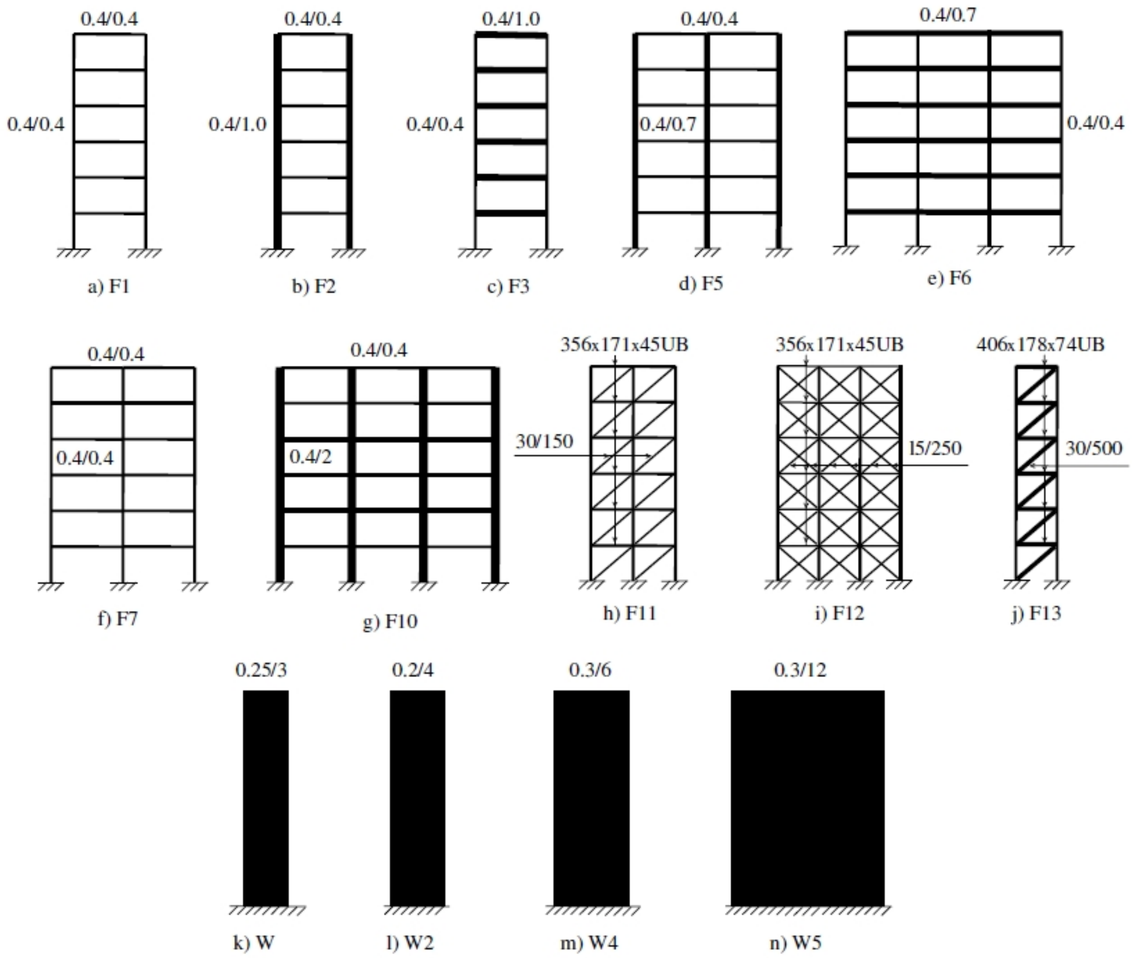

Fig. 7. Structures for the accuracy analysis. a)-g): reinforced concrete frames, h)-j): steel frames, k)-n): reinforced concrete shear walls.

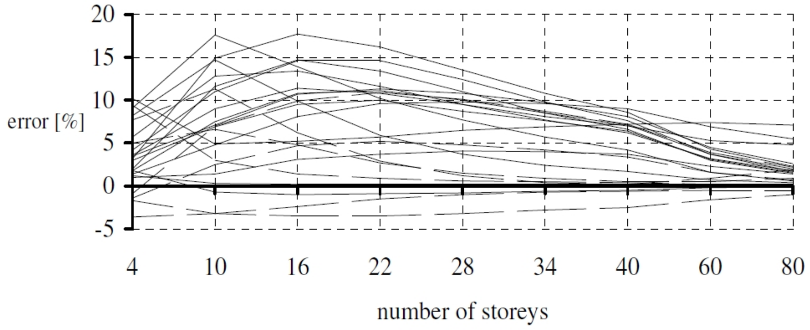

Fig. 8. Accuracy of "Solution 1: a simple method".

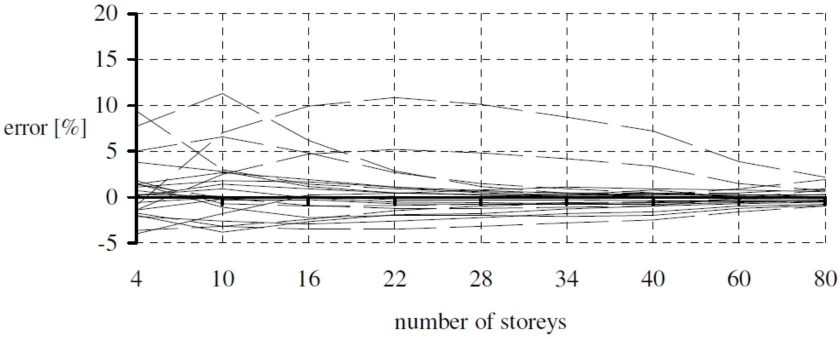

Fig. 9. Accuracy of "Solution 2: a more accurate method".

method" and "more accurate method") but, clearly, more information is needed if the proposed procedures are to be used for practical application.

In order to carry out a comprehensive accuracy analysis, 14 individual bracing units (F1, F2, F3, F5, F6, F7, F10, F11, F12, F13, W, W2, W4 and W5) were chosen whose details are given in Figure 7. Using these structures, twenty-two bracing systems were then created: F1W, F2-W, F3-W, F6-W5, F13-W2, F2-F5, F2-F5-W, F2-F5-F10, F2-F5-F10-W4, F3-F6, F3-F6-W2, F3-F6-F11, F3-F6-F11-W4, F1-F7, F1-F7-W2, F1-F6-F7, F1-F6-F7-W4, F1-F6-F7-F10, F1-F6-F7-F10-W4, F1-F6-F7-F12-F13, F1-F6-F7-F12-F13-W4 and F6-F10-W5. The height of the bracing units varied from 4 storeys to 80 storeys in nine steps. This resulted in 198 test structures. The storey height and the bays were 3 metres and 6 metres, respectively, in each case. The bracing units and systems were chosen to cover a wide range of structures. Among the bracing systems, there are bending dominated systems, shear dominated systems, mixed systems, systems consisting of frameworks only, systems consisting of frameworks and shear walls, systems consisting of reinforced concrete and steel bracing units, etc. The modulus of elasticity for the concrete and steel structures were $E = 25$ kN/mm^2 and $E = 200$ kN/mm^2, respectively.

The Finite Element based computer program Axis [2003] was used for the determination of the maximum deflection of the bracing systems and these results were considered "exact".

Figures 8 and 9 demonstrate the accuracy of the "simple method" and the "more accurate method", respectively. Solid lines represent systems with a shear wall and dashed lines mark systems that only contain frameworks.

In the case of the simple method, the error range proved to be -4% to +18%, with an average absolute error of less than 6%. Positive error means that the method overestimates the maximum deflection.

It is interesting to note that the simple method performs better when the bracing system does not contain shear walls. This follows from the fact that no significant (wall–frame) interaction is neglected.

The situation with the more accurate method is the opposite: as a rule, its performance is better when the bracing system also contains shear walls. This is a lucky coincidence as in practical situations the bracing system normally consists of frameworks and shear walls/cores. In such cases (solid lines in Figure 9) the error range of the more accurate method is quite spectacular: -4% to +4%, with a less than 1% average absolute error.

Compared to the "old" method [Zalka, 2009], both procedures proposed here are more accurate. The accuracy of the "simple solution" is slightly better (but the method itself is much simpler). The "more accurate solution" is still simpler and, as far as accuracy is concerned, spectacularly outperforms the "old" method.

7 Conclusions

In applying the continuum method to the deflection analysis of regular multi-storey buildings, it is not possible to create an equivalent column by simply adding up the characteristic stiffnesses of the bracing units in the hope of producing a simple and reliable solution as with the case of the stability and frequency analyses. However, it is possible to reduce the system of differential equations to the investigation of a single differential equation.

In doing so, two different avenues can be followed. In ignoring the direct interaction between the shear walls and frameworks, a very simple procedure can be produced.

Alternatively, when the direct interaction between the shear walls and frameworks is taken into account, a slightly more complicated but much more accurate solution can be produced for the deflection of the building. Based on the accuracy analysis of 126 test structures containing frameworks and shear walls/cores, its error range proved to be 4% to +4%, with a less than 1% absolute average error.

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