PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 46, NO. 2, PP. 239–242 (2002)

GENERALIZATION OF THE LAGRANGE INTERPOLATION POLYNOMIALS[1]

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Received: Jan. 25, 2002

Abstract

In this paper is to present generalization of the Lagrange interpolation polynomials in higher dimensions. The author gives this algorithm for numerical determine of this polynomials.

Keywords: multivariable interpolation.

1. Historical Review

More than two hundred years, in 1793 happened that the French mathematician J.L. LAGRANGE formulated the interpolation polynomial called later after him. He fitted on N points of the space $R^{(2)}$ a polynomial of $N-1$ degree and constructed it as a suitable linear combination of basic polynomials $L_{(i)}(x)$ giving in the i^{th} point the value 1, in the more point is zero.

The GAUSSIAN statement [1] is worthy of note: the LAGRANGIAN interpolation polynomial and the NEWTONIAN one are equivalent; but the first formulation has the advantage: its wanted polynomial can be written immediately, without solving a system of $N+1$ equations.

2. Univariable Lagrange Interpolation

As well known, this LAGRANGE interpolation is used systematically to produce the canonical form of a matrix function with one variable, if its minimal polynomial has simple roots only.

For numerical handling of a matrix function with more variables, e.g. to compute their canonical form, we have few knowledge, e.g. the infinite matrix power series are recommended [3], [4], [5], [6].

The treatment of stochastic processes – through systems of differential equations – raises the applicability of the LAGRANGE interpolation polynomials also in higher dimensions too. The author wishes to study and realize this possibility here, remarking that one cannot expect now the simplicity of the univariable case.

[1] In memoriam Dr. J. Egerváry

Let be given again N points in the space $R^{(n)}$ and we wish to fit onto these points a polynomial as a suitable linear combination of basic polynomials $L_{(i)}(\mathbf{x})$ with necessary demands:

$$L_{(i)}(\mathbf{x}_i) = 1, \quad L_{(i)}(\mathbf{x}_j) = 0 \quad \text{at } j \neq i.$$

3. Multivariable Lagrange Interpolation

If we search for a polynomial of F degree in $R^{(n)}$ and disregard the former demands, then the number n of the necessary points is:

$$p = \binom{n-1+F}{F}. \tag{1}$$

The polynomial fitted onto p points will not be LAGRANGIAN one. E.g. if $n = 3$, $F = 1$, so $p = 3$ and the second basic polynomial (with the notation $x(i) \equiv x_i$) shows the form:

$$\begin{aligned} H_2 = 0.25[&(x - x_1)/(x_2 - x_1) + (x - x_3)/(x_2 - x_3) \\ &+ (y - y_1)/(y_2 - y_1) + (y - y_3)(y_2 - y_3)], \end{aligned} \tag{2}$$

which does not give 0 value at the 1st and 3rd points. (The 3rd coordinate was regarded as a depending variable, furthermore the notation L_2 was not used.) E.g. if $n = 3$, $F = 2$, so the first basic polynomial, quadratic one appears as follows:

$$\begin{aligned} L_1(x, y) = 0.25[&(x - x_2)(x - x_3)/(x_1 - x_2)(x_1 - x_3) \\ &+ (x - x_2)(y - y_3)/(x_1 - x_2)(y_1 - y_3) + (x - x_3)(y - y_2)/(x_2 - x_3)(y_3 - y_2) \\ &+ (y - y_2)(y - y_3)/(y_1 - y_2)(y_1 - y_3)]. \end{aligned} \tag{3}$$

The other both basic polynomials can be written in analogous forms and all the 3 ones satisfy the LAGRANGIAN demands. Consequently, the wanted interpolation of the given function $z = z(\mathbf{x}) \equiv z(x, y)$ (with the notation $z_i \equiv z(\mathbf{x}_i)$) obtains this form:

$$z(x, y) \approx z_1 \cdot L_1(x, y) + z_2 \cdot L_2(x, y) + z_3 \cdot L_3(x, y) \tag{4}$$

and e.g. $z(x_2) = z_1 \cdot 0 + z_2 \cdot 1 + z_3 \cdot 0 = z_2$.

Obviously, the number of members in a basic polynomial is:

$$M = (n-1)^F. \tag{5}$$

4. Some Remarks to the Multivariable Lagrange Interpolation

Look at some rules to write the basic polynomials! Having N points in the space $R^{(n)}$ and the j^{th} coordinate of the i^{th} point: $x(i,j)$, one of k^{th} point: $x(k,j)$, then these coordinates possess the property: $x(i,j) \neq x(i,k)$ at $i \neq k$ (where $j = 1, 2, \ldots, n-1$; $i, k = 1, 2, \ldots, N$). The coordinates $x(i,j)$ will be the independent variables of the wanted interpolation's polynomial and the number of the basic polynomial too.

Mark with $u(j)$ the j^{th} coordinate for the running point of the wished basic polynomial and form with it the fraction:

$$[u(j) - x(j,k^*)]/[x(i,j) - x(j,k^*)] \tag{6}$$

at $k^* \neq k \in \{1, 2, \ldots, N\}$. By this formula, we can compute the elements for a matrix of N rows and $(N-1) \times (n-1)$ columns. Making e.g in the k^{th} row all products of F elements for $\forall k^* \neq k$, then we can obtain the known number $(n-1)^F$ of members in a basic polynomial. It is worth mentioning that the degree of the polynomial written by us is different from the traditional one of the polynomial fitted onto N points, because more demands are inflicted on the first one, namely its number is: $N(n-1)^F$. Just so many demands are necessary to write unambiguously N polynomials of F^{th} degree and $n-1$ variables.

Look at the following illustration:

Example. Fit a quadratic polynomial of two variables onto these points: $A(1, 2, 2)$, $B(2, 3, 15)$, $C(4, -2, -13)$. The three basic polynomials – following the sample of formula (3) – will be these:

$$L_1(x,y) = 0.25(1/3x^2 - 1/4y^2 + 1/12xy - 7/2x - 7/12y + 55/6),$$
$$L_2(x,y) = 0.25(-0.5x^2 + 0.2y^2 - 0.3xy + 3.9x + 1.8y - 7.2),$$
$$L_3(x,y) = 0.25(1/6x^2 + 0.05y^2 - 23/120xy - 0.05x + 1/15y - 1/15)$$

and with them the wanted polynomial is

$$z(x,y) \cong L(x,y) = 2L_1(x,y) + 15L_2(x,y) - 13L_3(x,y).$$

To use the basic polynomials in the mentioned applications, it is suitable to compute them by computer.

The author possesses the necessary programme and is ready to give it to each interested colleague.

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